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New Developments in Planning Accelerated Life Tests

Haiming Ma
Iowa State University

New developments in planning accelerated life tests

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Haiming Ma

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Program of Study Committee:
William Q. Meeker, Jr., Major Professor
Huaiqing Wu
Ranjan Maitra
Peng Liu
Xiaoli Tan

Iowa State University

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ABSTRACT

Accelerated life tests (ALTs) are often used to make timely assessments of the life time distribution of materials and components. The goal of many ALTs is estimation of a quantile of a log-location failure time distribution. Much of the previous work on planning accelerated life tests has focused on deriving test-planning methods under a specific log-location distribution. This thesis presents a new approach for computing approximate large-sample variances of maximum likelihood estimators of a quantile of general log-location distribution with censoring and time-varying stress based on a cumulative exposure model. This thesis also presents a strategy to develop useful test plans using a small number of test units.

We provide an approach to find optimum step-stress accelerated life test plans by using the large-sample approximate variance of the maximum likelihood estimator of a quantile of the failure time distribution at use conditions from a step-stress accelerated life test. In Chapter 2, we show this approach allows for multi-step stress changes and censoring for general log-location-scale distributions. As an application of this approach, the optimum variance is studied as a function of shape parameter for both Weibull and lognormal distributions. Graphical comparisons among test plans using step-up, step-down, and constant-stress patterns are also presented. The results show that, depending on the values of the model parameters and quantile of interest, each of the three test plans can be preferable in terms of optimum variance. In Chapter 3, using sample data from a published paper describing optimum ramp-stress test plans, we show that our approach and the one used in the previous work give the same variance-covariance matrix of the quantile estimator from the two different approaches. Then, as an application of this approach, we extend the previous work to a new optimum ramp-stress test plan obtained by simultaneously adjusting the ramp rate and the lower start level of stress. We find that the new optimum test plan can

have smaller variances than that of the optimum ramp-stress test plan previously obtained by adjusting only the ramp rate. We also compare optimum ramp-stress test plans with the more commonly used constant-stress accelerated life test plans.

Previous work on planning accelerated life tests has been based on large-sample approximations to evaluate test plan properties. In Chapter 4, we use more accurate simulation methods to investigate the properties of accelerated life tests with small sample sizes where large-sample approximations might not be expected to be adequate. These properties include the simulated bias and variance for quantiles of the failure-time distribution at use conditions. We focus on using these methods to find practical compromise test plans that use three levels of stress. We also study the effects of not having any failures at test conditions and the effect of using incorrect planning values. We note that the large-sample approximate variance is far from adequate when the probability of zero failures at certain test conditions is not negligible. We suggest a strategy to develop useful test plans using a small number of test units while meeting constraints on the estimation precision and on the probability that there will be zero failures at one or more of the test stress levels.

CHAPTER 1. GENERAL INTRODUCTION

1.1 Introduction

Accelerated life tests (ALT) have been widely used to estimate the lifetime of products in industry. When the lifetime of products at use conditions is much longer than the maximum permitted test time (which is almost always the case), engineers usually increase the levels of stresses (for example, temperature, voltage, humidity, or pressure) to higher than usual levels. They expect that at the higher levels of stress, the products will fail more quickly and that they can estimate the lifetime at use conditions using extrapolations based on an ALT model.

The ALT model usually has two components: a parametric distribution describing the failure-time distribution at fixed level of stress and a relationship between distribution parameters and levels of stresses. The relationship between life and stress is often proposed by engineers on the basis of physical or chemical theory. Then the engineers can use this relationship to extrapolate the lifetime of products at use level of stress based on parameters estimated from the life-time test at two or more accelerated levels of stress in the interval. If physical or chemical theory is not available, the engineers rely on previous experience to choose a model. If the levels of stress exceed a highest possible level of stress, the linear relationship may no longer hold. Exceeding this critical level of stress should be avoided in practice.

In the ALTs we have considered in this dissertation, we assume that only one accelerating stress is to be considered in designing a test plan and that the failure-time distributions are (log) location-scale distributions. The relationship, after a possible transformation, is linear between the location parameter of a log location distribution and a transformed level of stress within an interval of stress level bounded by the use level and another highest level at which testing will be permitted. We also assume that the scale parameter of the log-location-scale distribution does not depend on the level of stress.

There are several types of the ALT plans that have been proposed in the literature and that have been used in practice. Among them the most commonly used and extensively studied one is the constant-stress test plan. In a constant-stress test plan, the test units are allocated to two or more groups of different levels of stress. Usually, all the test units begin the test simultaneously and the test is run until a common censoring time. Recently, step-stress test plans have also drawn much attention. In a step-stress test plan, all the test units are tested at a common level of stress. The level of stress, however, can increase or decrease at points in time. If the change of stress levels is continuous, then we have a progressive-stress accelerated test plan. A progressive-stress AT plan with a constant rate of change leads to an important special case known as a ramp-stress test plan. In test plans other than constant-stress test plans, it is necessary to assume a cumulative exposure model that specifies how the probability of failure is affected as the level of stress changes over time. The cumulative exposure theory used in our research has been described in [3].

In designing a constant-stress test plan, engineers need to determine the allocations of test units to different groups and the level of stress for each group. In designing a step-stress or a ramp-stress test plan, engineers need to determine how the level of stress changes over time. In either case, before designing such a test plan, the engineers need information about the underlying distribution, the probability of failure of test units at two levels of stress. Such information, which is known as “planning information” or “planning values” of the parameters, is typically based on their previous experience with similar products or engineering judgment. The planning values are usually uncertain, which means that engineers usually only provide a range of the planning values that they believe contain the true underlying values.

The role of a statistician in designing an ALT plan is to help engineers with a specified amount of limited resources to get the most accuracy and precision from their experiment. Because maximum likelihood (ML) estimation is commonly used to analyze ALT data, a useful criterion for planning an ALT test is to minimize the variance of the ML estimator of some quantiles of a log location-scale distribution at the use conditions. This is usually done by comparing the large sample approximate variance of the estimators obtained from by inverting the Fisher information matrix for different test plans. If there were no uncertainty in our planning values, one could use an optimum test plan which, among all possible test plans, has the smallest variance of the estimator. In practice, however, there is always uncertainty in planning information and thus, as an alternative, it has been suggested that one should use a compromise test plan that has good (but not optimum) statistical properties, but that is also is robust to the uncertainty of the planning values.

A number of papers have been published to describe methods of designing optimum and compromise constant-stress, step-stress, and ramp-stress test plans using some well-known log location scale distributions, such as the exponential, Weibull and lognormal distributions. Most of the published work in this area has only aimed at analyzing the properties corresponding to a specific log location distribution and a specific type of test plans used in the papers. We know of no previous work that has, in these general cases, compared the variances of the ML estimators of a quantiles of a log location-scale distribution among constant-stress, step-stress and ramp-stress with censoring. At the same time, in practical applications, engineers want to know which type of test plans is more appropriate for their specific problem than others types of test plans. The research work presented in Chapters 2 and 3 are motivated by these kinds of questions.

Previous work on designing ALT plans has usually used the large-sample approximation approach. This approach is, however, questionable for the cases with a small numbers of test units. Yet, engineers are often constrained to use the smallest number of test units possible to make some kind of reasonable estimate of life at use conditions. The test plan designs obtained from large sample approximations applied to tests that actually have a small numbers of test units could be misleading. Studies along these lines have not been developed so far. The research work presented in Chapter 4 is motivated by these kinds of questions.

1.2 Overview of Available Literature

There is much literature describing research that has been done on ALT data analyses and on ALT plan designs. For example, Chapter 6 of Nelson [3] reviews constant-stress ALT plans. Chapter 10 of [3] describes step-stress and ramp-stress test plans and how to do data analyses, and gives references to the original sources. Chapters 18-20 of Meeker and Escobar [2] provided basic knowledge, detailed information and practical suggestions on ALT models, data analysis and test planning for constant-stress ALTs. Nelson [4, 5] provides an extensive bibliography describing much of the previous research work that has been done to study accelerated test plans. Escobar and Meeker [1] reviewed the ALT models that are commonly used in practice.

1.3 Dissertation Organization

This dissertation consists of three papers corresponding to Chapters 2, 3, and 4, respectively. Chapter 2 provides an approach for designing step-stress ALT plans for a general log-location-scale distribution. The approach includes the derivation of a general log likelihood for the associated cumulative exposure model, its derivatives with respect to distribution parameters, and expectations needed to compute the Fisher information matrix. Comparison of the large-sample approximate variance is presented among constant-stress, step-up-stress, and step-down-stress test plans under the same planning values for different log location-scale distributions.

Chapter 3 extends the approach of Chapter 2 to a limiting case with a continuously varying level of stress for a general log-location-scale distribution. We found that the values of the variance–covariance matrix from our general algorithm correspond to the value obtained from a different approach (only valid for Weibull distribution) proposed in a previously published paper. We also extend the one-dimensional optimum ramp-stress test plan that has been presented previously in the literature to a better two-dimensional optimum ramp-stress test plan. We also compare the new optimum test plan with the optimum constant-stress test plans under the same planning values.

Chapter 4 discusses how to choose a constant-stress test plan efficiently to meet a requirement of the precision in estimating lifetime of products when there are stringent constraints on the number of units that can be tested. We focus on practical three-level test plans because they are most commonly used in actual applications. We study the impact that not having any failures at test conditions will have on lifetime estimation. We then suggest and illustrate the use of a strategy combining both the conventional large-sample approximation approach and the more accurate simulation approach together. This strategy can be used to find a good test plan. In particular, these test plans have the smallest variance of the estimation while holding the probability of not having any failures at test conditions below a critical level and keeping the sample size as small as possible.

Chapter 5 provides overall conclusions based upon the results obtained in Chapters 2, 3 and 4. Two appendices are provided for the details in deriving formulas in Chapters 2 and 3, respectively.

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CHAPTER 2. OPTIMUM STEP-STRESS ACCELERATED LIFE TEST PLANS FOR LOG-LOCATION-SCALE DISTRIBUTION

Haiming Ma and W. Q. Meeker
Department of Statistics
Iowa State University
Ames, IA 50011

Abstract

This paper presents new tools and methods for finding optimum step-stress accelerated life test plans. First, we present an approach to calculate the large-sample approximate variance of the maximum likelihood estimator of a quantile of the failure time distribution at use conditions from a step-stress accelerated life test. The approach allows for multi-step stress changes and censoring for general log-location-scale distributions based on a cumulative exposure model. As an application of this approach, the optimum variance is studied as a function of shape parameter for both Weibull and lognormal distributions. Graphical comparisons among test plans using step-up, step-down, and constant-stress patterns are also presented. The results show that, depending on the values of the model parameters and quantile of interest, each of the three test plans can be preferable in terms of optimum variance.

Key Words – Cumulative exposure model, Large-sample approximate variance, Maximum likelihood.

Notation

s_U , s_H	pre-specified use stress and highest possible stress
$s_1, s_2, \ldots, s_h$	test stress levels
h	total number of stress levels in the experiment
ξ	standardized stress level
τ_i	time of stress change from s_i to s_{i+1} at each step, $i = 1,2,\ldots,h-1$
t, η	failure time and censoring time
μ, σ	location and scale parameters of a location-scale distribution
γ_0, γ_1	parameters of the log linear regression model
$\phi(\cdot)$, $\Phi(\cdot)$	pdf and cdf of a location-scale distribution
p_U , p_H	probabilities that a unit will fail by time η at use and the highest stress levels, respectively
z_p	p quantile of a standard location-scale distribution
$y_p = y_p(\xi)$	p quantile of a location-scale distribution at stress level ξ
n	total number of test units

2.1 Introduction

2.1.1 Accelerated Testing Background

In accelerated life tests (ALTs), units are tested at higher levels of stress (e.g., temperature, voltage, pressure) to obtain information about reliability in a small amount of time. ALTs are commonly used in product test plan processes (e.g., Nelson [10-14], Meeker and Escobar [8] and Escobar and Meeker [4]). Techniques for performing an ALT include constant stress (e.g., Chapter 6 of Nelson [14], Meeker and Hahn [7], Chapter 19 of Meeker and Escobar [8]) and step stress (e.g., Nelson [11], Chapter 10 of Nelson [12]).

In an ALT with constant stress, all the units are allocated into two or more groups to be tested separately at specified stress levels. When possible, the tests are run simultaneously to save time. In a step-stress ALT, units begin at a specified stress level. After a time fraction with respect to the total test length, the stress level is changed to another level. The stress level can be changed more than once before the end of the test. An advantage of step-stress ALTs is that increasing the level of stress during the test will generally result in failures happening more quickly. Another advantage of step-stress ALTs is that only one temperature-control chamber is needed. In a step-stress test all units are at a common temperature which changes at the step. In constant-stress ALTs there are usually separate chambers for each level of temperature.

Different choices of test-stress levels and unit allocations (for constant stress) or time fraction (for step stress) can result in different estimation precision of a quantity of interest such as a quantile of the life distribution at use conditions. Our goal is to compare test plans in terms of the optimum (i.e., the smallest) variance of maximum likelihood (ML) estimates and to provide insight and tools that can be used to find a practical, statistically efficient accelerated test plan.

2.1.2 Previous work in planning accelerated tests

A number of research papers have developed methods for optimum and robust ALT test plans by suitably choosing test length, levels of accelerating variables and allocation of test units. It is often assumed that the purpose of an ALT is to estimate a quantile of the failure distribution at use conditions. The criterion for choosing an ALT plan is often to find a plan that gives minimum variance of the ML estimator of a particular quantile. In this section we describe related previous work on planning step-stress experiments. Within the framework of Nelson's cumulative exposure model [12], Miller and Nelson [9] first presented theory for optimum test plans for a single-step stress test with complete failure data (i.e., all units run to failure) from an exponential distribution. Bai, Kim and Lee [2] extended the theory of Miller and Nelson to censoring. Their work was further extended to optimum multi-step step-stress by Khamis and Higgins [5] and Yeo and Tang [16] and to type II censoring under the exponential distribution by Xiong [15].

Optimum multi-step-stress test plans have also been developed using the Weibull and lognormal distributions with censoring. Bai and Kim [3] developed an approach to multi-step stress for the Weibull distribution with the highest stress level at the last step. More recently, Alhadeed and Yang [1] obtained optimum step-stress plans for complete failure data using the lognormal distribution.

Miller and Nelson [9] compared constant and step-stress test plans for the exponential distribution. Their results showed that step-stress tests yielded the same amount of information as constant-stress tests under the exponential distribution with complete failure data. They optimized their step-stress plan by choosing the time of the step that minimizes the approximate variance of the ML estimator of a quantile of interest in their test plans.

Khamis [6] compared constant stress and step-stress test plans for the Weibull distribution with a known shape parameter and fixed lower stress. His results showed that step-stress test plans may have an advantage over constant-stress test plans when there is heavy censoring for the constant-stress test at lower stress levels.

Although much work has been done to find optimum step-stress ALT plans, the tradeoffs between constant and step-stress testing in terms of variance of ML estimation of quantiles of the failure time distributions has not been clearly evaluated. Within the step-stress test plans, focus has been on the step-up because it is thought to assure failures quickly.

Step-down test plans also have potential value. The idea is that initial aging of units at high stress could provide more failures at lower levels of stress and thus better information when the test is continued at lower levels of stress. Also, previous research for step-stress ALTs has been for particular distribution like the exponential, Weibull and lognormal distributions. We extend Bai and Kim's results [3] to provide a more general theory and methods for multiple-step stress ALTs for the log location-scale family of distributions. The optimization is done with respect to both positions of the steps and stress fractions in step-stress, simultaneously. The different properties among constant, step-up-stress, and step-down-stress test plans are presented graphically with respect to the optimum variance of failure-quantile estimators.

2.1.3 Overview

In section 2.2, we describe three test plans that are used in this paper. In section 2.3, we present the model that we use to minimize the large-sample approximate variance of the ML estimator of a specified quantile of a log location-scale distribution in a general multi-step-stress ALT. Section 2.4 shows the optimum variance for a simple step-stress test as a function of the shape parameter of the Weibull and lognormal distributions for different combinations of the planning values. This section also presents comparisons between simple step-stress and constant-stress test plans. Section 2.5 provides some concerns in practical applications including compromise test plans and an assessment of the effect of incorrectly specifying the planning value for σ. Section 2.6 gives some simulation results to check the adequacy of the large-sample approximate variances. Section 2.7 draws some conclusions and outlines areas for future research in this area.

2.2 Test Plans

In this paper, we compare the optimum large-sample approximate variances of the ML estimators among three basic test plans. In most ALT models there is an implied transformation of stress (e.g., log of voltage). We use s to denote this transformed stress. All tests will be between the use stress s_U and the highest possible stress s_H. Under the three test plans, test units are tested at the pre-specified highest stress level s_H and at another stress level s_L where $s_U \le s_L < s_H$. For convenience, in the following we use the standardized stress $\xi = (s - s_U)/(s_H - s_U)$, where $0 \le \xi \le 1$. Thus $\xi_U = 0$, $\xi_H = 1$. The maximum test length is η.

The three test plans compared in this paper are

(1) Constant-stress test plan

When only two stress levels are involved, all of the test units are divided into two groups. One group is tested at ξ_H and the other group is tested at ξ_L. Both groups are tested simultaneously from the beginning until the censoring time η. When three stress levels are involved, all of the test units are divided into three groups. One group is tested at ξ_H, another group is tested at ξ_L and the third group is tested at a stress being the middle of ξ_L and ξ_H. Three groups are tested simultaneously from the beginning until the censoring time η.

(2) Step-up stress test plan

When only two stress levels are involved, all of the test units begin the test at ξ_L. At time τ_1 $(\tau_1 < \eta)$, all of the surviving units are moved to ξ_H and tested until time η. This is called a *simple step-up stress test plan*. When three stress levels are involved, all of the test units begin the test at ξ_L. At time τ_1 $(\tau_1 < \eta)$, all of the surviving units are moved to a stress being middle of ξ_L and ξ_H until τ_2 $(\tau_1 < \tau_2 < \eta)$. Then all of the surviving units are moved to ξ_H and tested until time η.

(3) Step-down stress test plan

When only two stress levels are involved, all of the test units begin the test at ξ_H. At time τ_1 $(\tau_1 < \eta)$, all the surviving units are moved to ξ_L and tested until time η. This is known as a *simple step-down stress test plan*. When three stress levels are involved, all of the test units begin the test at ξ_H. At time τ_1 $(\tau_1 < \eta)$, all of the surviving units are moved to a stress midway between ξ_L and ξ_H and tested until τ_2 $(\tau_1 < \tau_2 < \eta)$. Then all of the surviving units are moved to ξ_L and tested until time η.

When only two stress levels are involved, by changing ξ_L and the allocation of the test units at ξ_L simultaneously for constant-stress test plans or by changing ξ_L and τ_1 simultaneously for step-stress test plans, one can obtain the optimum test plans characterized by the large-sample approximate variance of the ML estimators. When three stress levels are involved, we consider a case with a fixed allocation of the test units (for constant) and fixed time duration $\tau_2 - \tau_1$ (for step-stress) at the middle level of stress. Then the optimum test plans can be obtained by changing ξ_L and the allocation of the test units at ξ_L simultaneously for constant-stress test plans or by changing ξ_L and τ_1 simultaneously for step-stress test plans.

2.3 The Model and Log Likelihood

2.3.1 Model

We assume that at any level of stress the failure time T follows a log location-scale distribution with constant σ and cdf

$$\Pr[T \le t; \mathrm{T}] = \Phi\left[\frac{\log(t) - \mu}{\sigma}\right],$$

where the location parameter is $\mu = \gamma_0 + \gamma_1 \xi$. We also assume that Nelson's [12] cumulative exposure model holds. This model implies that the distribution of remaining life of a test unit depends only on the cumulative exposure it has received no matter how it was exposed. Let

$z_i(t)$, the standardized log time at stress level *i*, be defined as $z_i = [\log(t-\delta_{i-1}) - \mu(\xi_i)]/\sigma$, $i = 2,3,\ldots,h$, and $z_\eta = [\log(\eta-\delta_{h-1}) - \mu(\xi_h)]/\sigma$, where $\tau_{i-1} \le t \le \tau_i$ and τ_i is the time of stress change from s_i to s_{i+1} with $\tau_0 = 0$ and h is the total number of stress levels in the experiment.

The time shift induced by the change in stress levels from the beginning of the test to the beginning of step *i* is

$$\delta_{i-1} = \tau_{i-1} - \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1}) e^{\gamma_1(\xi_i - \xi_j)}.$$

Note that δ_{i-1} is positive for a step-up stress test plan and negative for a step-down stress test plan, because γ_1 is typically negative. Under the cumulative exposure model, we have $z_i(\tau_i) = z_{i+1}(\tau_i)$, $i = 2,3,\ldots,h$.

The failure probabilities at the highest stress ($\xi_H = 1$) and the lowest stress ($\xi_U = 0$) are: $p_H = \Phi\left(\dfrac{\log(\eta) - \gamma_0 - \gamma_1}{\sigma}\right)$ and $p_U = \Phi\left(\dfrac{\log(\eta) - \gamma_0}{\sigma}\right)$, respectively. It is easy to express the failure probability at any value of ξ as a function of p_U and p_H.

2.3.2 Log likelihood

The likelihood for a single test unit having a log location-scale failure time distribution under a multi-step-stress test is:

$$\mathrm{L} = \prod_{i=1}^{h} \left[\frac{1}{\sigma(t-\delta_{i-1})} \phi(z_i) \right]^{U_i(t)} [1 - \Phi(z_\eta)]^{U_{h+1}(t)},$$

where $U_i(t) = 1$ if t is within the step of ξ_i and zero otherwise, $i = 1,2,\cdots\cdots h$. $U_{h+1}(t) = 1$ if t is larger than η and zero otherwise. The log likelihood for a single test unit is

$$l = \sum_{i=1}^{h} U_i(t) \{-\log(\sigma) - \log(t-\delta_{i-1}) + \log[\phi(z_i)]\} + U_{h+1}(t) \log[1 - \Phi(z_\eta)]. \qquad (1)$$

The first and the second partial derivatives of (1) with respect to the model parameters are given in the appendix. The total log likelihood is obtained by summing (1) over all test units. The ML estimators $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{\sigma}$ are those values that maximize the total log likelihood.

2.3.3 The large-sample approximate variance

In models that meet standard regularity conditions (including the log location scale distributions used in this paper), the Fisher information matrix can be used to quantify the expected information that an experiment will provide on a set of parameters. Also, the large-sample approximate variance-covariance matrix of the ML estimators is the inverse of the Fisher information matrix. Details are presented in the appendix.

Our goal is to minimize the large-sample approximate variance of the ML estimator of the p quantile of the log failure time distribution at the use stress. The ML estimator of the p quantile at stress level ξ can be expressed as

$$\hat{y}_p = \hat{\gamma}_0 + \hat{\gamma}_1 \xi + z_p \hat{\sigma}.$$

The large-sample approximate variance of $\hat{y}_p$ is

$$\text{Avar}(\hat{y}_p) = (1, \xi, z_p) \Sigma_{\hat{\gamma}_0, \hat{\gamma}_1, \hat{\sigma}} (1, \xi, z_p)^T,$$

where Avar is used to denote the large sample approximate variance of a scalar quantity, $\Sigma_{\hat{\gamma}_0, \hat{\gamma}_1, \hat{\sigma}}$ is the large-sample approximate variance-covariance matrix given in the appendix, and the superscript ^{T} indicates vector transpose. Our objective is to minimize $\text{Avar}(\hat{y}_p)$ at $\xi = 0$. The test plan properties to be optimized are τ_i and ξ_i, $i = 1, 2, \ldots, h-1$. The optimization can be done using the function optim() in R. To solve the problem of multiple optima, multiple start values are necessary to achieve the minimum variance in the optimization.

For comparing the relative efficiency of test plans with different samples sizes we can use the scaled large-sample approximate variance defined as $(n/\sigma^2)\text{Avar}(\hat{y}_p)$. The large-sample approximate variance-covariance matrix and properties of optimum test plans depend only on the planning values p_H, p_U and σ. This is because given p_H, p_U, σ and a censoring time η, one can calculate γ_0 and γ_1, providing a complete specification of the model and the amount of censoring for the test plan. These planning values are usually obtained from previous experience with a similar product or from engineering judgment.

2.4 Comparison of Simple Step-Stress and Constant-Stress Plans

2.4.1 Comparison of approximate variance

Figures 2.1 and 2.2 show the value of scaled variance $(n/\sigma^2)\text{Avar}(\hat{y}_p)$ obtained from optimum ALT plans with two stress levels as a function of σ for three different quantiles of interest: 0.01, 0.10 and 0.50. These figures also compare constant-stress, step-up-stress, and step-down-stress test plans for different values of p_H and p_U under the Weibull and lognormal distributions, respectively. Some important conclusions from these figures are:

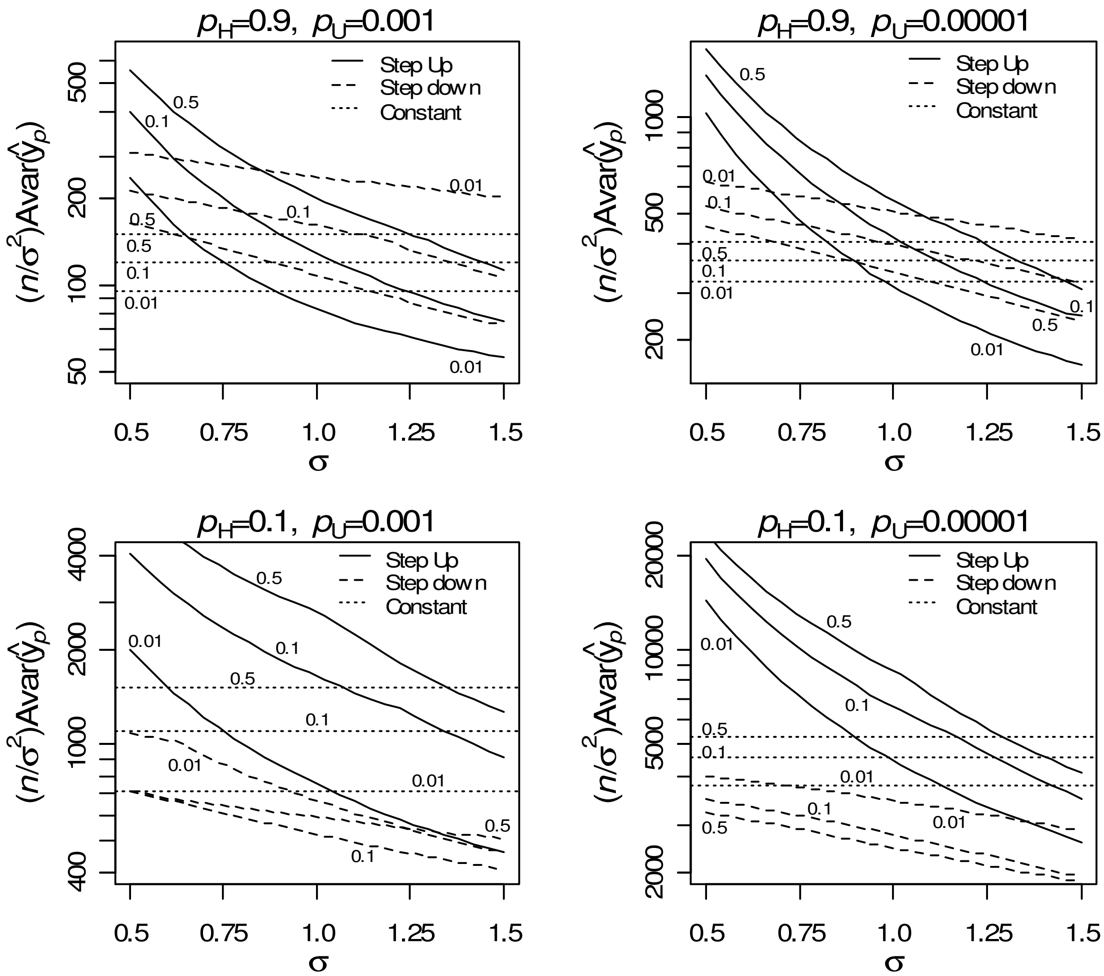

Figure 2.1: Scaled optimum variance as a function of σ for the Weibull distribution at three quantile of interest for four combinations of planning values under three test plans. The three quantiles of interest are *p* = 0.01, 0.1 and 0.5, respectively. The four combinations of planning values are (p_H = 0.9, p_U = 0.001), (p_H = 0.9, p_U = 0.00001), (p_H = 0.1, p_U = 0.001), and (p_H = 0.1, p_U = 0.00001), respectively. The three test plans investigated are step-up stress, step-down stress and constant stress, respectively.

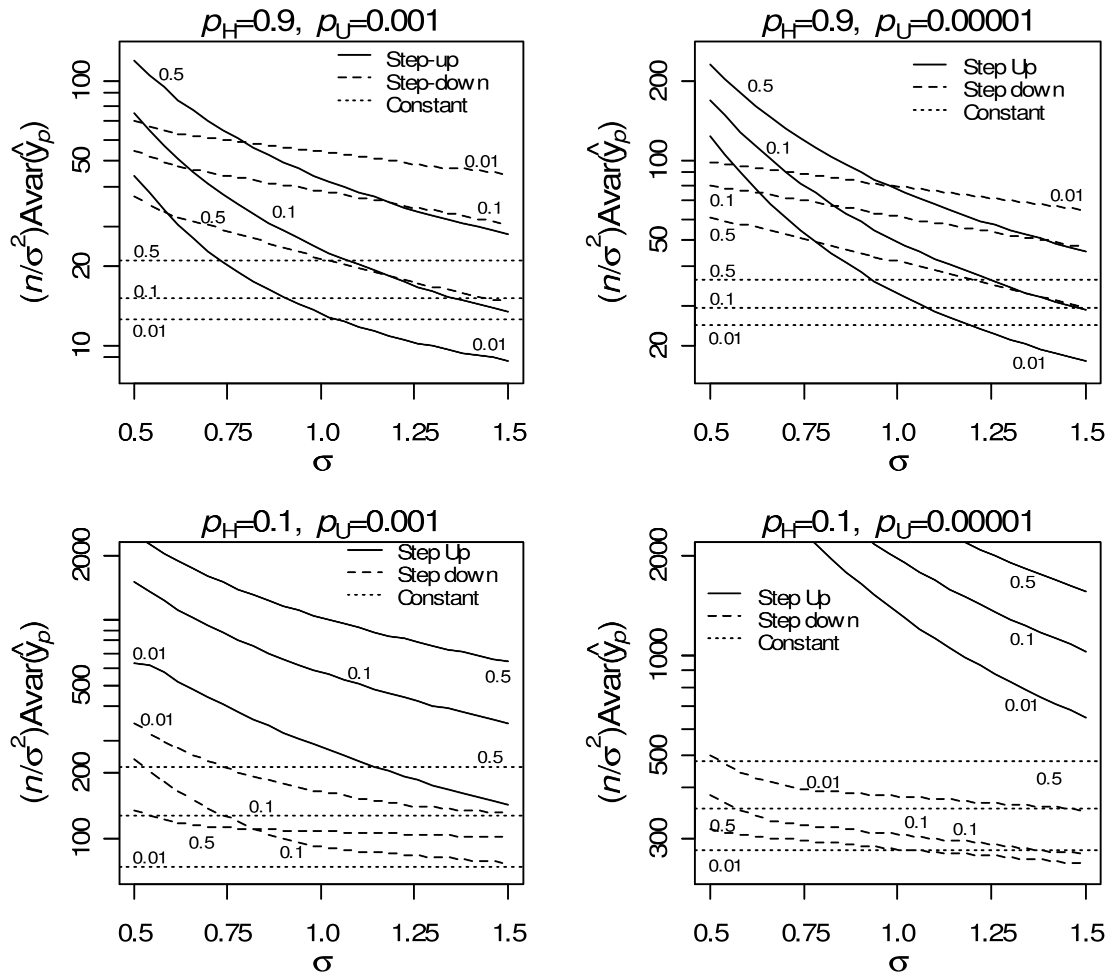

Figure 2.2: Scaled optimum variance as a function of σ for the lognormal distribution at three quantile of interest for four combinations of planning values under three test plans. The three quantiles of interest are p = 0.01, 0.1 and 0.5, respectively. The four combinations of planning values are (p_H = 0.9, p_U = 0.001), (p_H = 0.9, p_U = 0.00001), (p_H = 0.1, p_U = 0.001), and (p_H = 0.1, p_U = 0.00001), respectively. The three test plans investigated are step-up stress, step-down stress and constant stress, respectively.

a) In the step-up and step-down test plans, the scaled optimum variance usually decreases as σ increases. As it is known from previous work (e.g., Nelson and Kielpinski [12]), the scaled variance for constant-stress plans does not depend on σ.

b) Step-up plans are usually poor relative to step-down plans when p_H is small. Depending on the planning values, step stress plans can have either a smaller or a larger optimum variance than constant stress plans. For plans with small σ and large p_H, the constant-stress plans are usually better than the step-stress plans.

c) Given the same values of p_H and p_U, the optimum variance of the 0.10 quantile ML estimator is usually larger than that of the 0.01 quantile and smaller than that of the 0.50 quantile in a step-up plan. Constant-stress plans have the same ordering as that of step-up stress plans. Interestingly, the curves for a step-down plan have the reverse order. However, this reverse order can change when p_H and p_U approach each other as shown in the SW corners of Figures 2.1 and 2.2.

d) Comparing test plans with the same values of p_H and p_U, the test plans under the lognormal distribution have a smaller optimum variance than those for the corresponding Weibull distributions. This is because a lognormal distribution has a lighter lower tail than that of the corresponding Weibull distributions with the same first-two moments. Thus, with the same amount of change in the lower tail of the distribution, the corresponding change the log quantile y_p of the lognormal distribution is smaller than that of the Weibull distribution. Note also that the relationship between the shape parameter and the variance of ML estimators of quantiles of a Weibull distribution is different from that of a lognormal distribution (see, e.g., [8], page 83). This fact needs to be considered when comparing the optimum variances between the two distributions with the same planning values p_H and p_U.

2.4.2 Comparison of Test Plan Parameters

Besides the optimum variance shown in Figures 2.1 and 2.2, it is also interesting to look at the time fraction and the lower level of stress under the optimum conditions. Figure 2.3 shows examples of the time fraction of the first step and the lower stress for 0.01 quantile with the step-up test plan and 0.5 quantile with the step-down test plan under the Weibull distributions. The reason for choosing these two quantiles is that they can provide smaller variances than that of the corresponding quantiles of a constant-stress test plan in some interval of σ values, as seen in Figure 2.1.

Figure 2.3 shows that the time fraction and lower stress change as σ changes under the optimum step-stress test plans. Concerning the possible uncertainty of the planning value of σ, one would prefer that the change of the time fraction and lower stress be as small as possible within the range of the σ uncertainty. Note that a constant-stress plan does not depend on σ. Under certain circumstances, it is possible that a substantial change in the time fraction or the lower stress may not lead to a large change in $\mathrm{Avar}(\hat{y}_p)$, if this function is very flat around the optimum point. In practice, one should assess the robustness of step-stress test plans relative to the uncertainty in σ.

The change of the time fraction and lower stress as a function of σ is not always smooth like in Figure 2.3. In some situations, these functions can have a jump change across certain σ values due to the multiple optima of the objective variance function.

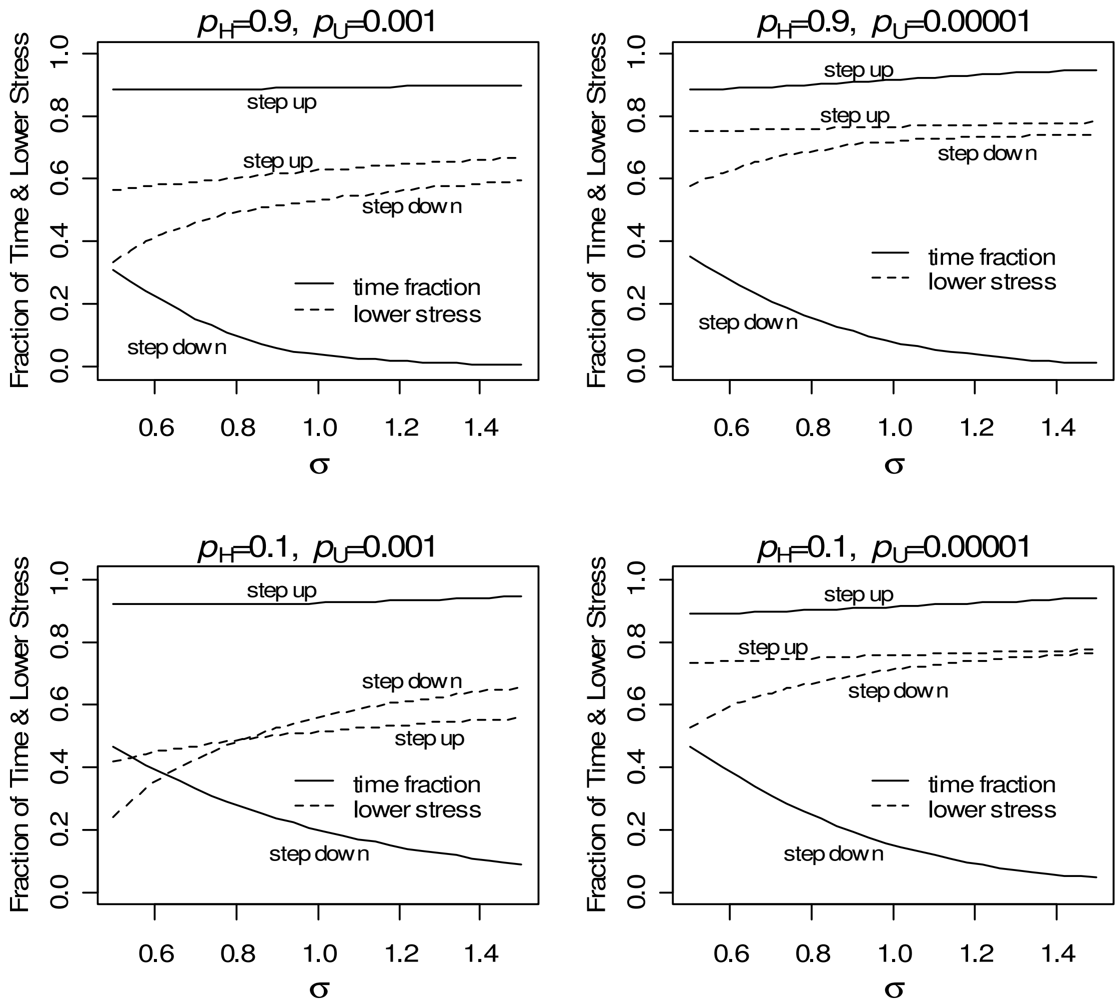

Figure 2.3: The optimum time fraction of the first step and the lower stress for the 0.01 quantile of step-up plan and the 0.5 quantile of step-down plan for the Weibull distribution as a function of σ under the optimum step-stress test plans.

2.5 Some Concerns in Practical Applications

2.5.1 Avoiding plans with a small expected number of failures

The optimum time fraction and lower stress under the optimum conditions of step-stress test plans are obtained under the assumption of an infinitely large sample. In some cases, the time fraction of a step of stress can be close to zero, which means that the optimum test plans are impractical. This is because for a large-sample approximate variance to provide an adequate approximation, one needs to have a sufficiently large number of failures in each step of stress. If a step is too short, a very large number of sample units will be required to generate a sufficiently large number of failures. To avoid this problem, one can use a constraint: the time fraction of each step must be within, say, 5% and 95% of the total test time. A similar constraint also can be applied to the standardized lower stress. The actual values of the constraints should be determined in each specific situation.

In the rest of this paper, we use the 5%-95% constraint for both time fraction and lower stress. With this constraint, we reproduced results like those in Figures 2.1 and 2.2. These results (details not given here) show that imposing such constraints may, as expected, increase the variance somewhat. For example, at $\sigma = 1.5$, without the constraint, the optimum variances of the 0.1 quantile for step-down under the Weibull distribution are 86.91 and 272.82 for $p_U = 0.001$ and 0.00001, respectively. With the constraint, the corresponding optimum variances increase to 96.17 and 277.41 for $p_U = 0.001$ and 0.00001, respectively.

2.5.2. Effect of misspecification of the σ planning value

Section 2.4.2 indicates that different σ planning values will result in different optimum test plans. Thus, it is interesting to investigate the change in $\text{Avar}(\hat{y}_p)$ due to the misspecification of the σ planning value**.**

Figure 2.4 shows the ratio $\text{Avar}(\hat{y}_p)/\text{Avar}_{\text{OPT}}$ on a log scale as a function of σ for the Weibull distribution and four different sets of possible true failure probabilities under the fixed optimum step-stress test plans with Avar_{OPT} being $\text{Avar}(\hat{y}_p)$. The left side of Figure 2.4 is for a step-up plan for the 0.01 quantile optimized at p_H = 0.9, p_U = 0.001 and $\sigma =$ 1.25. The right side of Figure 2.4 is for a step-down plan for the 0.5 quantile optimized at p_H = 0.1, p_U = 0.00001 and $\sigma = 0.75$. For simplicity, we assume the uncertainty range of the planning values of p_H, p_U and σ are [$p_H/10$, min($10p_H$, 1)], [$p_U/10$, min($10p_U$, p_H)] and [$\sigma - 0.25$, $\sigma + 0.25$], respectively. Figure 2.4 provides four curves for each test plan corresponding to four possible true parameters: (p_H, p_U), ($p_H/10$, p_U), (p_H, $p_U/10$) and ($p_H/10$, $p_U/10$). The four straight horizontal lines are the relative variances of the corresponding optimum constant-stress plans for the four situations. Because our primary concern is the variance increment, the cases involving $10p_H$ and $10p_U$ are omitted because they have smaller variance than those involving $p_H/10$ and $p_U/10$.

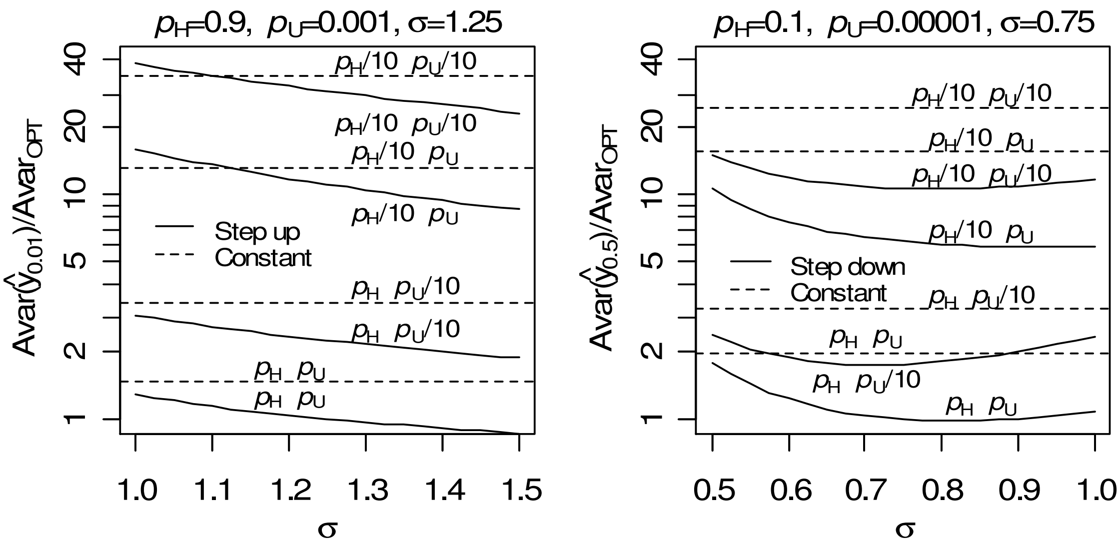

Figure 2.4: The ratio $\mathrm{Avar}(\hat{y}_p)\big/\mathrm{Avar}_{\mathrm{OPT}}$ on a log scale as a function of σ for the Weibull distribution and four different sets of possible true failure probabilities under fixed optimum step-stress test plans with $\mathrm{Avar}_{\mathrm{OPT}}$ being $\mathrm{Avar}(\hat{y}_p)$. On the left side is a step-up test plan for the quantile *p* = 0.01 optimized at p_H = 0.9, p_U = 0.001 and σ = 1.25. On the right side is a step-down test plan for the quantile *p* = 0.5 optimized at p_H = 0.1, p_U = 0.00001 and σ = 0.75. The straight horizontal lines are the relative variances of the corresponding optimum constant-stress plans. The labels on the lines indicate the perturbed true values of the failure probabilities.

Figure 2.4 demonstrates that it is useful to assess robustness to uncertainty in σ. Whether uncertainty in σ seriously affects the test plan or not depends on each specific situation. Figure 2.4 shows that, for some values of σ, optimum step-stress test plans can have smaller variance than that of the corresponding optimum constant stress plans. If, however, there is a very large amount of uncertainty in σ, a constant-stress plan might be preferable.

2.5.3 Compromise test plans

Test plans with three stress levels are often used to gain robustness to the uncertainty of planning values and verify the linear relationship of the location and the standardized stress. Meeker and Hahn [7] proposed a 4:2:1 compromise constant-stress test plan that has been widely applied, where 4, 2 and 1 represent the relative proportions of unit allocations at the low, middle and high levels of stresses, respectively.

In this section, we investigate the relationship between the minimum value of $\left(n/\sigma^2\right)\text{Avar}\,(\hat{y}_p)$ and σ under compromise test plans involving three stress levels. For constant-stress plans, we follow Meeker and Escobar (chapter 20 of [8]) and constrain 15% of the units to be tested at a point halfway between the highest level of stress and the lowest level of stress. The lowest level of stress and the unit allocation at this level are optimized with respect to the large sample approximate variance. Without such a constraint, in any practical situation when optimizing, the 3-level plan will degenerate to a 2-level plan. For step-stress compromise test plans with h = 3, we use a similar constraint, where 15% of the test time is spent at a level of stress halfway between the highest level of stress and the lower level of stress. The lowest level of stress and the test time at this level are optimized using the variance objective function in Section 2.3.3. Again, without such a constraint, in any practical situation, an optimum $h \geq 3$ plan will degenerate to an h = 2 plan. Note that the 5-95% constraint described in Section 2.5.1 for unit allocation, time fraction and lower stress still hold. For simplicity, the step stress plans considered here are either straight step up or straight step down in stress.

Figures 2.5 and 2.6 show the value of scaled variance $\left(n/\sigma^2\right)\text{Avar}\,(\hat{y}_p)$ obtained from optimum ALT plans involving three stresses as a function of σ for three different quantiles of interest: 0.01, 0.10 and 0.50, respectively, for the Weibull and lognormal distributions. These figures also compare constant-stress, step-up-stress, and step-down-stress test plans for different values of p_H and p_U under the Weibull and lognormal distributions, respectively. Comparing with the results in Figures 2.1 and 2.2, the scaled variance for the test plans with three stress levels have the σ dependence similar to that involving just two stress levels.

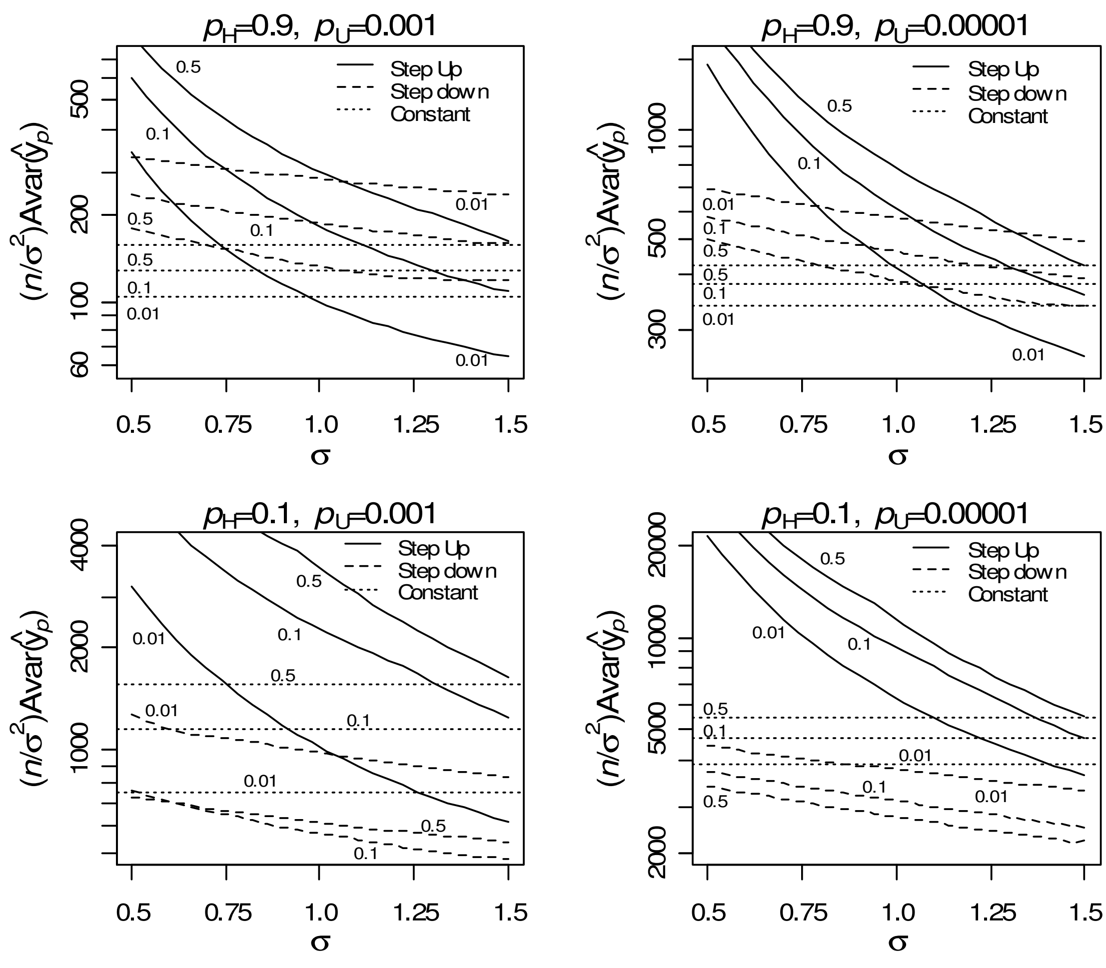

Figure 2.5: Scaled optimum variance as a function of σ for the Weibull distribution at three quantiles of interest for four combinations of planning values under three test plans. The three quantiles of interest are p = 0.01, 0.1 and 0.5, respectively. The four combinations of planning values are (p_H = 0.9, p_U = 0.001), (p_H = 0.9, p_U = 0.00001), (p_H = 0.1, p_U = 0.001) and (p_H = 0.1, p_U = 0.00001), respectively.

The three test plans investigated are the three-stress-level step-up stress, step-down stress and constant stress, respectively.

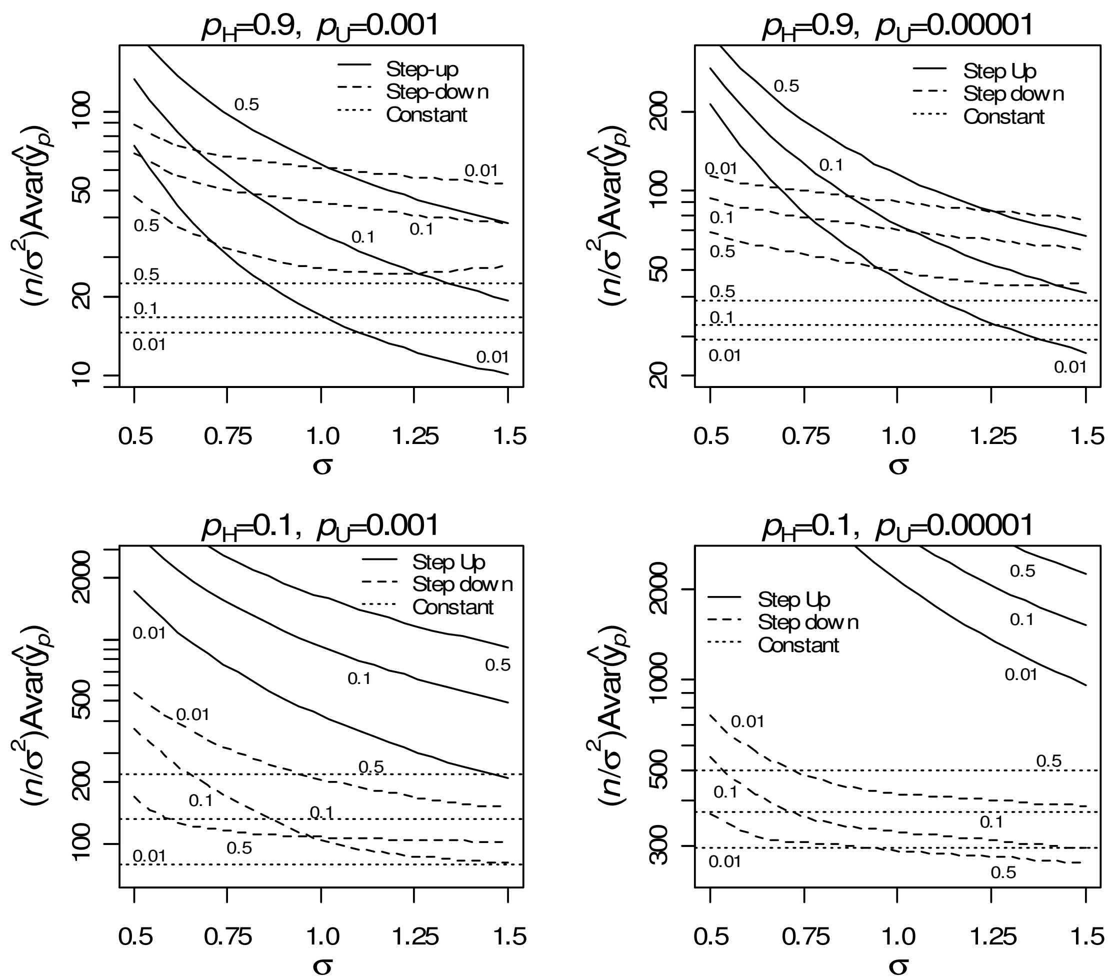

Figure 2.6: Scaled optimum variance as a function of σ for the lognormal distribution at three quantiles of interest for four combinations of planning values under three test plans. The three quantiles of interest are *p* = 0.01, 0.1 and 0.5, respectively. The four combinations of planning values are (p_H = 0.9, p_U = 0.001), (p_H = 0.9, p_U = 0.00001), (p_H = 0.1, p_U = 0.001) and (p_H = 0.1, p_U = 0.00001), respectively. The three test plans investigated are the three-stress-level "compromise" step-up stress, step-down stress and constant stress, respectively.

2.6 Adequacy of the Large Sample Approximate Variance

In practical applications, simulation is frequently used to complement evaluations done with large-sample approximations (e.g., Chapter 6 of [12] or Chapter 20 of [8]) to evaluate test plans design. Here we provide some results of such simulations.

Figure 2.7 shows the simulated scaled variance as a function of the expected total number of failures under four optimum stress-test test plans, compared with the large-sample approximation. As expected, when the expected total number of total units gets large, the simulated variance becomes close to the large-sample approximate variance. The results are presented in terms of the expected number of failures instead of sample size because, as is true in other similar situations involving censored data, the adequacy of the large-sample approximation is approximately a function of the expected number of failures, rather than the actual sample size (notice that the shapes of the functions in Figure 2.7 are all similar).

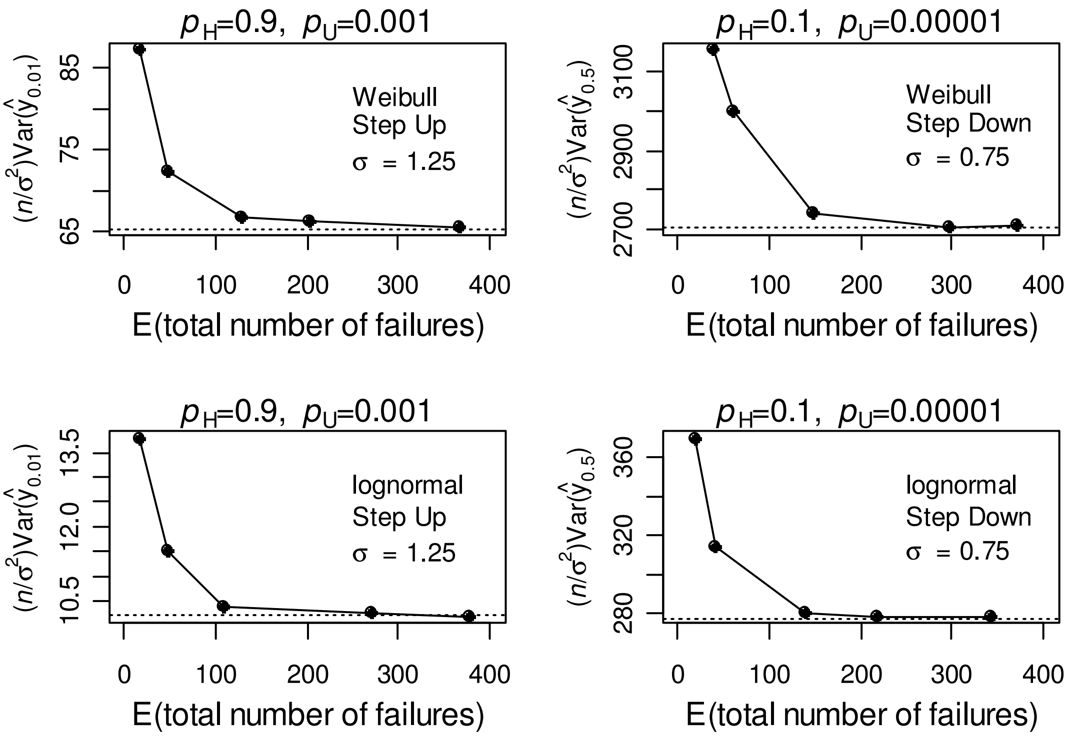

Figure 2.7: Simulated scaled variance as a function of the expected total number of failures under optimum stress-test test plans (large dots with solid lines for visual guidance) compared with the large-sample approximation (horizontal dashed line). The number of simulations at each point is 20,000.

While simulation methods are useful, the amount of computer time needed to do an evaluation of a particular test plan will take orders of magnitude longer that the large-sample approximate variance method. For example, given planning values p_H , p_U and σ to be 0.1, 0.001 and 0.8, respectively, and a plan with 1000 test units, it took approximately 42 seconds to evaluate a particular step-up test plan under the Weibull distribution on a typical desktop computer with a 2.4GHz processor. The analytical large sample approximation can be computed more than 1000 times faster. For this reason, an extensive study of optimum test plans is not really practicable when using simulation to do evaluations. On the other hand, graphical presentation of simulation results can generally give a much better approximation to the true variances (limited only by Monte Carlo error) and provide detailed information and additional insight about test plan properties as well as the verification of the analytical approach.

2.7 Concluding Remarks and Areas for Future Research

This paper extends previous work that has been done on planning step-stress ALTs. We give a general approach for computing the large-sample approximate variance of ML estimators of quantiles of interest under the widely used log-location-scale family of distributions for multi-step-stress ALTs with censoring. The approach allows comparison among different test plans under different assumed log location-scale distributions.

Comparison of step-stress test plans shows that the scaled optimum variance decreases as σ increases. In terms of the scaled optimum variance, the simple step-down test plan usually performs better than the simple step-up test plan when the failure probability at the level of the highest stress is small.

Compared with step-stress test plans, the above results show that, when both time/allocation fraction and lower stress are optimized, constant-stress plans are a good choice when σ is small and p_H is not too small. This is because constant-stress plans usually provide test plans with a smaller optimum variance of the ML estimators. Step stress plans can have a smaller optimum variance than constant stress plans when σ is large or when p_H is very small (as it often is in practice).

Although the numerical results in this paper are focused on the comparison among simple constant-stress test plans and step-stress test plans, the approach given here provides a way to evaluate and compare a variety of more complicated multi-step-stress test plans for any log-location-scale distribution. Possible areas for future research include

- Extension to some other non-log-location-scale distributions and other time-dependent stress levels than step stress under general log-location-scale distributions.
- Exploration of other step-stress test plans not studied in this paper that might provide robustness to the uncertainty of planning values or the possible departure from log linear location-stress model.
- Application of step-stress methods to accelerated tests that provide degradation data.

Appendix

This appendix provides the derivatives of the log likelihood described in Section 2.3 and the details of the approach that we used to calculate the large-sample approximate variance-covariance matrix of the ML estimators of γ_0, γ_1 and σ. For a log-location-scale distribution, we have the following properties

$$\frac{\partial z_i}{\partial \gamma_0} = -\frac{1}{\sigma}, \ \frac{\partial z_i}{\partial \sigma} = -\frac{z_i}{\sigma} \text{ and } \frac{\partial z_i}{\partial \gamma_1} = \frac{1}{\sigma}(\nu_i - \xi_i) \ ,$$

where $\nu_i = \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1})(\xi_i - \xi_j) e^{\gamma_1(\xi_i - \xi_j)} / (t - \delta_{i-1})$, $i = 2,3,\ldots,h$. Let $\nu_1(t) = 0$ and ν_η is ν_h at $t = \eta$. z_i is defined in Section 2.2. The step-up test plan results in a positive ν_i, while the step-down results in a negative ν_i; for constant stress, $\nu_i = 0$, $i = 2,3,\ldots,h$. This leads to dramatic differences in the properties of optimum ALT among these three test plans.

By means of the above properties, for a log-location-scale distribution, the first partial derivatives with respect to parameters of the log likelihood for a single observation can be expressed as

$$\frac{\partial l}{\partial \gamma_0} = -\frac{1}{\sigma}\left[\sum_{i=1}^{h} U_i(t)\frac{1}{\phi(z_i)} \cdot \frac{d\phi(z_i)}{dz} - U_{h+1}(t)\frac{1}{1-\Phi(z_\eta)} \cdot \frac{d\Phi(z_\eta)}{dz}\right]$$

$$\frac{\partial l}{\partial \gamma_1} = -\frac{1}{\sigma}\left\{\sum_{i=1}^{h} U_i(t)\left[\sigma v_i - \frac{1}{\phi(z_i)}\cdot\frac{d\phi(z_i)}{dz}\left(v_i - \xi_i\right)\right] + U_{h+1}(t)\frac{1}{1-\Phi(z_\eta)}\cdot\frac{d\Phi(z_\eta)}{dz}\left(v_\eta - \xi_h\right)\right\}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{1}{\sigma}\left\{\sum_{i=1}^{h} U_i(t)\left[1 + \frac{z_i}{\phi(z_i)}\cdot\frac{d\phi(z_i)}{dz}\right] - U_{h+1}(t)\frac{z_\eta}{1-\Phi(z_\eta)}\cdot\frac{d\Phi(z_\eta)}{dz}\right\}.$$

Define

$$A(z) = \frac{1}{\phi(z)}\cdot\frac{d^2\phi(z)}{dz^2} - \frac{1}{\phi^2(z)}\cdot\left(\frac{d\phi(z)}{dz}\right)^2,\ B(z) = \frac{1}{1-\Phi(z)}\cdot\frac{d^2\Phi(z)}{dz^2} + \frac{1}{\left[1-\Phi(z)\right]^2}\cdot\left(\frac{d\Phi(z)}{dz}\right)^2.$$

Note that the first partial derivatives are zero when evaluated at ML estimates when they are not on the boundary of the parameter space. We set terms corresponding to the first partial derivatives equal to zero in the following expressions for the second partial derivatives of the log likelihood. The resulting second partial derivatives of the log likelihood for a single observation, evaluated at the ML estimates are:

$$\frac{\partial^2 l}{\partial \gamma_0^{\,2}} = \frac{1}{\sigma^2}\left[\sum_{i=1}^{h} U_i(t)A(z_i) - U_{h+1}(t)B(z_t)\right].$$

$$\frac{\partial^2 l}{\partial \gamma_1^{\,2}} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)\left[-\sigma^2 v_i' + \sigma v_i'\frac{1}{\phi(z_i)}\cdot\frac{d\phi(z_i)}{dz} + (v_i - \xi_i)^2 A(z_i)\right]\right\}$$

$$-\frac{U_{h+1}(t)}{\sigma^2}\left[\sigma v_\eta'\frac{1}{1-\Phi(z_\eta)}\cdot\frac{d\Phi(z_\eta)}{dz} + \left(v_\eta - \xi_h\right)^2 B(z_\eta)\right]$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)\left[-1 + z_i^{\,2}A(z_i)\right] - U_{h+1}(t)z_\eta^{\,2}B(z_\eta)\right\}$$

$$\frac{\partial^2 l}{\partial \sigma\,\partial \gamma_0} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)z_iA(z_i) - U_{h+1}(t)z_\eta B(z_\eta)\right\}$$

$$\frac{\partial^2 l}{\partial \gamma_1\partial \gamma_0} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)\cdot\left(\xi_i - v_i\right)A(z_i) + U_{h+1}\left(v_\eta - \xi_h\right)B(z_\eta)\right\}$$

$$\frac{\partial^2 l}{\partial \gamma_1\partial \sigma} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)\left[-\sigma v_i - \left(v_i - \xi_i\right)\cdot z_iA(z_i)\right] + U_{h+1}(t)\left(v_\eta - \xi_h\right)\cdot z_\eta B(z_\eta)\right\},$$

where $\nu_i' = \partial \nu_i / \partial \gamma_1 = \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1})(\xi_i - \xi_j)^2 e^{\gamma_1(\xi_i - \xi_j)} / (t - \delta_{i-1}) - \nu_i^2$, $\nu_\eta' = \nu_h'(\eta)$ and $i = 2,3,\ldots,h$.

Let $\theta = (\gamma_0, \gamma_1, \sigma)$. The Fisher information matrix of a step-stress ALT can be computed as $I_\theta = E\left[-\frac{\partial^2 l}{\partial\theta\,\partial\theta^T}\right]$. After factoring out the common factor of $1/\sigma^2$, F still depends on σ through $\partial^2 l/\partial\gamma_1^2$, $\partial^2 l/\partial\gamma_0\partial\gamma_1$ and $\partial^2 l/\partial\gamma_1\partial\sigma$, a phenomenon different from the constant-stress ALT (e.g., Nelson and Kielpinski [10]).

Defining $C = \frac{1}{\phi(z)}\cdot\left(\frac{d\phi(z)}{dz}\right)^2 - \frac{d^2\phi(z)}{dz^2}$, $D = \frac{d\phi(z_\eta)}{dz} + \frac{\phi(z_\eta)^2}{1-\Phi(z_\eta)}$,

$R_m = \int_{-\infty}^{z_i} x^m C(x)dx$, $G_{1,i} = \int_{z_{i-1}}^{z_i} \left(\lambda_i e^{-\sigma x} - k_i^2 e^{-2\sigma x}\right)\phi(x)dx$, $G_{2,i} = \int_{z_{i-1}}^{z_i} \left(\lambda_i e^{-\sigma x} - k_i^2 e^{-2\sigma x}\right)\frac{d\phi(x)}{dx}dx$,

$H_{m,i} = \int_{z_{i-1}}^{z_i} e^{-m\sigma x} C(x)dx$, $Q_{m,i} = \int_{z_{i-1}}^{z_i} e^{-m\sigma x} xC(x)dx$ and $W_i = \int_{z_{i-1}}^{z_i} e^{-\sigma x}\phi(x)dx$ with $i = 1,2,\ldots,h$, $m =$ 0, 1 or 2, and $z_0 = -\infty$, one has

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_0^2}\right) = R_0 + D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_1^2}\right) = \sum_{i=1}^{h}\left[\sigma^2 G_{1,i} - \sigma G_{2,i} + k_i^2 H_{2,i} - 2k_i\xi_i H_{1,i} + \xi_i^2 H_{0,i}\right] + \sigma\nu_\eta'\phi(z_\eta) + (\nu_\eta - \xi_h)^2 D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\sigma^2}\right) = \Phi(z_\eta) + R_2 + z_\eta^2 D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\sigma\,\partial\gamma_0}\right) = R_1 + z_\eta D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_1\partial\gamma_0}\right) = \sum_{i=1}^{h}\left[\xi_i H_{0,i} - k_i H_{1,i}\right] + (\xi_h - \nu_\eta)D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_1\partial\sigma}\right) = \sum_{i=1}^{h}\left[\sigma\cdot k_i W_i + \xi_i Q_{0,i} - k_i Q_{1,i}\right] + (\xi_h - \nu_\eta) z_\eta D$$

where $k_i = \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1})(\xi_i - \xi_j) e^{-\gamma_0 - \gamma_1 \xi_j}$, $\lambda_i = \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1})(\xi_i - \xi_j)^2 e^{-\gamma_0 - \gamma_1 \xi_j}$ with $i = 2,3,\ldots,h$ and $k_1 = \lambda_1 = 0$.

In the case of the Weibull distribution, $C = e^z \phi_{\text{sev}}(z)$ and $D = \phi_{\text{sev}}(z_\eta)$, where $\phi_{\text{sev}}(z) = \exp[z - \exp(z)]$ is the standard pdf of extreme value distribution. For the lognormal distribution, $C = \phi_{\text{nor}}(z)$ and $D = \dfrac{\phi_{\text{nor}}^2(z_\eta)}{1 - \Phi_{\text{nor}}(z_\eta)} - z_\eta \phi_{\text{nor}}(z_\eta)$, where $\phi_{nor}(z) = \left(1/\sqrt{2\pi}\right)\exp\left(-z^2/2\right)$ is the standard pdf of normal distribution.

Under the standard regularity conditions, the large-sample approximate variance-covariance matrix of the ML estimators of γ_0, γ_1 and σ is

$$\Sigma_{\hat\gamma_0,\hat\gamma_1,\hat\sigma} = \begin{bmatrix} A\,\text{var}(\hat\gamma_0) & A\,\text{cov}(\hat\gamma_0,\hat\gamma_1) & A\,\text{cov}(\hat\gamma_0,\hat\sigma) \\ A\,\text{cov}(\hat\gamma_0,\hat\gamma_1) & A\,\text{var}(\hat\gamma_1) & A\,\text{cov}(\hat\gamma_1,\hat\sigma) \\ A\,\text{cov}(\hat\gamma_0,\hat\sigma) & A\,\text{cov}(\hat\gamma_1,\hat\sigma) & A\,\text{var}(\hat\sigma) \end{bmatrix}$$

$$= \frac{1}{n} \begin{bmatrix} E\left(-\dfrac{\partial^2 l}{\partial \gamma_0{}^2}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \gamma_1}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \sigma}\right) \\ E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \gamma_1}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_1{}^2}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_1 \partial \sigma}\right) \\ E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \sigma}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_1 \partial \sigma}\right) & E\left(-\dfrac{\partial^2 l}{\partial \sigma^2}\right) \end{bmatrix}^{-1}.$$

For specific log location-scale distributions including Weibull and lognormal, the above equations may be further simplified. Simplifying the equations may be useful for the purpose of computing efficiency. However, in our calculation the difference of computational time or precision is negligible in practice between using the above formulas and using the formulas of special cases (e.g., those in [3]) for step-up test plans of the Weibull distribution.

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CHAPTER 3. A TOOL FOR EVALUATING ACCELERATED LIFE TEST WITH TIME-VARYING STRESS USING A LOG LOCATION-SCALE DISTRIBUTION

Haiming Ma and W. Q. Meeker
Department of Statistics
Iowa State University
Ames, IA 50011

Abstract

Accelerated life tests (ALTs) are often used to make timely assessments of the life time distribution of materials and components. The goal of many ALTs is estimation of a quantile of a log-location failure time distribution. Much of the previous work on planning accelerated life tests has focused on deriving test-planning methods under a specific log-location distribution. This paper presents a new approach for computing approximate large-sample variances of maximum likelihood estimators of a quantile of general log-location distribution with censoring and time-varying stress. The approach is based on a cumulative exposure model. Using sample data from a published paper describing optimum ramp-stress test plans, we show that our approach and the one used in the previous work give the same variance-covariance matrix of the quantile estimator from the two different approaches. Then, as an application of this approach, we extend the previous work to a new optimum ramp-stress test plan obtained by simultaneously adjusting the ramp rate and the lower start level of stress. We find that the new optimum test plan can have a smaller variance than that of the optimum ramp-stress test plan previously obtained by adjusting only the ramp rate. We also compare optimum ramp-stress test plans with the more commonly used constant-stress

accelerated life test plans. We also conduct simulations to provide insight and to check the adequacy of the large-sample approximate results obtained by the approach.

Key Words – Cumulative exposure model, Large-sample approximate variance, Maximum likelihood.

Notation

V_L, V_U, V_H	Initial low level, pre-specified use level, and highest possible level of the original stress
s	Transformed stress. When voltage is the ramp-stress $s = \log(V_H / V)$
ξ	Standardized stress $\xi = (s - s_U)/(s_H - s_U)$
τ	Time to reach the highest possible level of stress $\tau = (V_H - V_L)/k$
t, η	Failure time and censoring time
μ, σ	Location and scale parameters of a location-scale distribution
γ_0, γ_1	Parameters of the log linear regression model when $\mu = \gamma_0 + \gamma_1 \xi$
γ_0', γ_1'	Parameters of the log linear regression model when $\mu = \gamma_0' + \gamma_1' s$
w	$w(t) = \int_0^t \exp(-\gamma_1 \xi(x))\, dx$
α	V_H / V_U
$\phi(\cdot)$, $\Phi(\cdot)$	pdf and cdf of a location-scale distribution
p_U, p_H	Probabilities that a unit will fail by time η at use and the highest stress levels, respectively
z_p	p quantile of a standard location-scale distribution
$y_p = y_p(\xi)$	p quantile of a location-scale distribution at stress level ξ
n	Total number of test units

3.1. Introduction

3.1.1 Accelerated Testing Background

Accelerated life tests (ALT) are commonly used in product design processes. Because there is limited time to launch new products, engineers use accelerated tests to obtain needed information on the reliability by raising the levels of certain acceleration variables like temperature, voltage, humidity, stress, and pressure. Techniques for performing an ALT include constant stress, step stress, and ramp-stress, among others. Evaluation of the variance of an estimator of a log location-scale distribution quantile (e.g., Weibull or lognormal) with varying stress has a lot of practical applications. Statisticians help engineers design statistically efficient ALT plans and assess estimation precision as a function of sample size. Reviews of the research work in this area can be found in, for example, Nelson [12-15], Meeker and Hahn [7], and Meeker and Escobar [8].

Most previously developed methods to calculate the large sample approximate variance of Maximum Likelihood (ML) estimators of distribution quantiles treated only a particular log location-scale distribution. For example, Miller and Nelson [10] presented a theory for optimum step-stress test plans for an exponential distribution. Bai and Kim [4] developed a step-stress theory of the large-sample approximate variance for the Weibull distribution. Alhadeed and Yang [1] give an optimum simple step-stress plan for the lognormal distribution.

In addition to the development for individual log location-scale distributions, it is useful to develop an approach for computing the large sample approximate variance for a general log-location-scale distribution. Besides the elegance of having just one algorithm, the generalization allows evaluations for log-location-scale distributions beyond the more commonly used exponential, Weibull, and log-normal distributions, including the loglogistic and Fréchet distributions (e.g. Chapter 4 of Meeker and Escobar [8]). Recently, Ma and Meeker [6] provided an approach to calculate the large sample approximate variance in step-stress test plans for a general log location-scale distribution. In this paper, we adapt the results of Ma and Meeker [6] to the case of a test plans with the stress varying continuously over time and show how to use the results to find ramp-stress test plans that have good statistical properties.

3.1.2 Ramp-stress accelerated test background

Ramp-stress ALTs have been used in practice (see, for example, Chapter 10 of Nelson [13]). Yin and Sheng [16] studied the properties of the ML estimator of the exponential life distribution parameter from a ramp-stress test plan. Nelson [10, Chapter 10] presented theory to calculate the ML estimator of a quantile associated with a ramp-stress ALT using a Weibull distribution and a particular cumulative exposure model. Bai, Cha and Chun [2] considered ramp-stress ALTs with two ramp rates for the Weibull distribution under Type-I censoring. Bai, Chun and Cha [3] developed a method for finding an optimum ramp-stress ALT test plan for a Weibull distribution by choosing a ramp rate to minimize a large-sample approximate variance.

In a ramp-stress ALT, units begin the test at a low level of stress and the level of stress increases linearly over time at a constant rate. The test plan can be adjusted by the starting stress level and the rate, given the use and highest levels of stress, and a censoring time. There are two ways to specify test termination. One is that the level of stress increases linearly until a specific censoring time. The other is that initially the level of stress increases linearly. After a time fraction of the censoring time, the level of stress reaches the pre-specified highest level. Then the highest level of stress will be applied to the surviving test units until the specific censoring time.

A ramp-stress ALT can be viewed as a limit of a multiple step-up stress ALT when the number of steps approaches infinity while the change in stress level at each of step approaches zero and the sum of the step jumps is held constant. Like step-up stress ALTs, ramp-stress ALTs can result in failures happening more quickly by increasing the ramp rate and such tests only require a single temperature-controlled chamber for testing.

Different choices of the stress start level and the ramp rate can affect estimation precision of a quantity of interest such as a quantile of the life distribution at use conditions. Our goal is to find test plans with the optimum (i.e., the smallest) variance of ML estimators of such quantiles.

3.1.3 Overview

The rest of this paper is organized as follows. Section 3.2 describes the ramp-stress test plans used in this paper. Section 3.3 presents the time-varying stress model and likelihood for a general log-location-scale life distribution. This section also shows how to compute the large-sample approximate variance of the ML estimator of a specified quantile of the life distribution under the general time-varying model. Section 3.4 obtains optimum ramp-stress plans that minimize the large-sample approximate variance of the ML estimator of a specified quantile and compares these optimum plans with previously-suggested optimum ramp-stress plans and optimum constant-stress plans in terms of the approximate scaled large-sample variance. This section also gives the results of a simulation to check the adequacy of the large-sample approximate variances. Section 3.5 states some conclusions and outlines areas for future research.

3.2 Test Plans

In most ALT models there is an original stress scale (e.g., voltage) and an implied one-to-one transformation of stress (e.g., log of voltage). We use *V* and *s* to denote the original and transformed stresses, respectively. The stress applied to test specimens in a ramp-stress test will always be between an initial lower level of stress V_L and the highest possible level of stress V_H. V_L can be either higher or lower than the use level of stress V_U. As in [3], in the ramp stress test plans we consider *V* as a voltage and use a transformed stress, $s = \log(V_H / V)$. Sometimes, for convenience, a standardized stress $\xi = (s - s_U)/(s_H - s_U)$ is used. This standardized stress has the nice properties that $\xi_U = 0$ and $\xi_H = 1$. The maximum test length is η time units (e.g., hours). If we define the time for the stress level to reach V_H to be τ, then $\tau = (V_H - V_L)/k$ and $\tau/\eta = k_0/k$, where $k_0 = (V_H - V_L)/\eta$ is the constant rate of change in *V* (e.g., volts per hour).

Figure 3.1 shows the two possible test schemes. (a) $\tau \le \eta$, i.e., $k \ge k_0$. (b) $\tau > \eta$, i.e., $k < k_0$. By simultaneously optimizing V_L and k, one can usually obtain a smaller variance of the quantile estimator by using scheme (a) , relative to scheme (b).

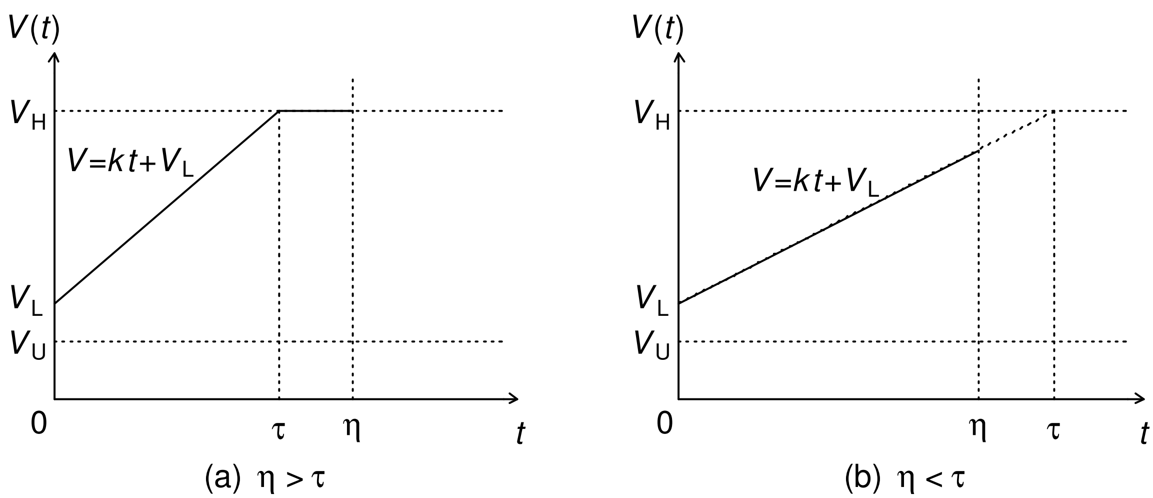

Figure 3.1: Ramp-stress test schemes considered in this paper.

3.3 The Model and Log Likelihood

3.3.1 Model

This section shows how to compute the large-sample approximate variance of the ML estimator of a quantile of a general log-location-scale distribution at use conditions from continuously time-varying stress accelerated life tests. We begin with a multiple step-stress test. Then we let the number of steps approach infinity as the change of stress level at each step approaches zero, holding the sum of the step jumps constant.

We assume that at any level of stress the failure time T follows a log location-scale distribution with cdf

$$\Pr[T \leq t; \mathrm{T}] = \Phi\left[\frac{\log(t) - \mu}{\sigma}\right],$$

where the location parameter is $\mu = \gamma_0 + \gamma_1 \xi$ and the scale parameter σ is constant. We also assume that Nelson's [13] cumulative exposure model holds. This model implies that the distribution of remaining life of a test unit depends only on the cumulative exposure it has received no matter how it was exposed.

In a multiple step-stress test, we define the standardized log time at stress level *i*, to be $z_i = [\log(t - \delta_{i-1}) - \mu(\xi_i)]/\sigma$, $i = 2,3,\ldots,h$, and $z_\eta = [\log(\eta - \delta_{h-1}) - \mu(\xi_h)]/\sigma$, where $\tau_{i-1} \leq t \leq \tau_i$, τ_i is the time at which the stress level change from s_i to s_{i+1}, $\tau_0 = 0$, and h is the total number of stress levels in the experiment.

The time shift induced by the change in stress levels from the beginning of the test to the beginning of step *i* is

$$\delta_{i-1} = \tau_{i-1} - \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1}) e^{\gamma_1(\xi_i - \xi_j)}.$$

Under the cumulative exposure model, we have $z_i(\tau_i) = z_{i+1}(\tau_i)$, $i = 2,3,\ldots,h$. When $h \to \infty$, under the limiting process described at the beginning of this section, we have $z(t) = [\log(w(t)) - \gamma_0]/\sigma$, where $w(t) = \int_0^t \exp(-\gamma_1 \xi(x))\,dx$. The failure probabilities at the highest level of stress ($\xi_H = 1$) and the use level of stress ($\xi_U = 0$) can be expressed as $p_H = \Phi\left(\dfrac{\log(\eta) - \gamma_0 - \gamma_1}{\sigma}\right)$ and $p_U = \Phi\left(\dfrac{\log(\eta) - \gamma_0}{\sigma}\right)$, respectively.

3.3.2 Log likelihood

The log likelihood for a single test unit having a log location-scale failure time distribution under a multi-step-stress test is:

$$l = \sum_{i=1}^{h} U_i(t)\left\{-\log(\sigma) - \log\left(t - \delta_{i-1}\right) + \log\left[\phi(z_i)\right]\right\} + U_{h+1}(t)\log\left[1 - \Phi(z_\eta)\right]. \tag{1}$$

where $U_i(t)$ = 1 if t is within the step of ξ_i and $U_i(t)$ = 0 otherwise, $i = 1,2,\cdots\cdots h$. $U_{h+1}(t)$ = 1 if t is larger than η and $U_{h+1}(t)$ = 0 otherwise. When $h \to \infty$, under the limiting process described in this section, the log likelihood is

$$l = U_1(t)\{-\log(\sigma) - \gamma_1 \xi(t) - \log(w(t)) + \log[\phi(z)]\} + U_2(t)\log\left[1 - \Phi(z_\eta)\right], \tag{2}$$

where $U_1(t)$ = 1 if $t < \eta$ and $U_1(t)$ = 0 otherwise. $U_2(t)$ = 0 if $t < \eta$ and $U_2(t)$ = 1 otherwise. The total log likelihood is obtained by summing (2) over all test units. The ML estimators $\hat{\gamma}_0$, $\hat{\gamma}_1$, and $\hat{\sigma}$ are those values that maximize the total log likelihood. The first and the second partial derivatives of (2) with respect to the model parameters are given in the appendix.

3.3.3 The large-sample approximate variance

Under the standard regularity conditions (which hold for log-location-scale distributions), $\Sigma_{\hat{\gamma}_0, \hat{\gamma}_1, \hat{\sigma}}$, the large-sample approximate variance-covariance matrix of $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{\sigma}$, the ML estimators of the model parameters, is the inverse of the Fisher information matrix (FIM). The FIM is the expectation, with respect to the data, of the negative Hessian matrix [the second derivatives of log likelihood in (2) with respect to the model parameters], evaluated at the model parameters. The appendix provides expressions for the elements of the FIM.

The ML estimator of the *p* quantile at standardized stress ξ is $\hat{y}_p = \hat{\gamma}_0 + \hat{\gamma}_1 \xi + z_p \hat{\sigma}$. The large-sample approximate variance of $\hat{y}_p$ is $\text{Avar}(\hat{y}_p) = (1, \xi, z_p) \Sigma_{\hat{\gamma}_0, \hat{\gamma}_1, \hat{\sigma}} (1, \xi, z_p)^T$, where the superscript ^{T} indicates vector transpose. Our goal is to minimize $\text{Avar}(\hat{y}_p)$ at $\xi = 0$ (i.e., at the use conditions). The test plan properties to be optimized are V_L and *k*. We did the optimization by using the function optim() in R [11].

To compare the relative efficiency of test plans with different samples sizes we use the scaled large-sample approximate variance defined as $(n/\sigma^2)\text{Avar}(\hat{y}_p)$. The large-sample approximate variance-covariance matrix and properties of optimum test plans depend on the planning values of the parameters p_H, p_U and σ. As usual when dealing with locally optimum designs (i.e., when the optimum depends on the model parameters), such planning values are obtained from some combination of previous experience and engineering judgment.

In the case of the ramp-stress test plans we consider, $s(t) = \log(V_H/(kt + V_L))$ when $kt + V_L \le V_H$ and $s(t) = 0$ otherwise. Note that ramp stress refers to a test in which $V(t)$ is linear in *t*. $s(t)$ will not, in general, be linear in *t*. Simple deduction shows that $\alpha = V_H/V_U$ defines the shape of time dependence of $s(t)$ and thus $\xi(t)$. Therefore, in addition to p_H, p_U

and σ, the large-sample approximate variance-covariance matrix and properties of the optimum test plans will also depend on α.

3.4 Optimum Ramp-Stress Plans

3.4.1 Comparison with existing optimum ramp stress plans

In [3], a one-dimensional optimum ramp-stress test plan is proposed. This plan is similar to that given in Figure 3.1 except that V_L = 0 and the ramp rate is optimized. The location parameter is expressed as $\mu = \gamma_0^{'} + \gamma_1^{'} s$, which is a little different from our notation described in Section 3.3. It is easy to convert from one parameterization to the other. Bai, Chun and Cha [3] considered a case where $\gamma_0^{'}$ = 6.0, $\gamma_1^{'}$ = 9.0, σ = 0.5, V_L = 0 kV, V_U = 20 kV, V_H = 40 kV and η = 2400 seconds. The quantile of interest is p = 0.1. Bai, Chun and Cha [3] also provided a simulated sample data yielding estimates $\hat{\gamma}_0^{'}$ = 6.13, $\hat{\gamma}_1^{'}$ = 8.73, $\hat{\sigma}$ = 0.451. Our approach gives the same variance-covariance with respect for ($\hat{\gamma}_0^{'}$, $\hat{\gamma}_1^{'}$, $\hat{\sigma}$) as that given in [3], where the authors used a different theoretical approach that is only valid for the Weibull distribution.

Using $\gamma_0^{'}$ = 6.0, $\gamma_1^{'}$ = 9.0 and σ = 0.5, as input to the algorithm proposed in this paper we find that the one-dimensional optimum $\left(n/\sigma^2\right)\text{Avar}\left(\hat{y}_{0.1}\right)$ is 1632 at k = 24.0 V/sec when V_L is fixed at 0 kV. We extend the one-dimensional optimization of the ramp-stress test plan in Bai, Chun and Cha [3] into a two-dimensional optimization. By selecting k = 18.9 V/sec and V_L = 13.9 kV, we obtain the optimum $\left(n/\sigma^2\right)\text{Avar}\left(\hat{y}_{0.1}\right)$ = 1493, a nearly 9% reduction in variance with respect to that of the one-dimensional optimization. In the rest of this section, we will only work with the optimum test plan from our two-dimensional optimization.

3.4.2 Comparison of the large sample approximate variance with simulated variance

To assess the adequacy of the large-sample approximate variances used in this paper, we conducted simulations to compare with the actual variance as a function of expected total numbers of failures during the test. Figure 3.2 shows the simulated scaled variance as a function of the expected total number of failures under the two-dimensional optimum ramp-stress test plan. As expected, when the expected total number of failures gets large, the simulated variance approaches the large-sample approximate variance. In this test plan, the probability that a test unit will fail during the test is 99.99%. Thus the expected total number of failures is almost exactly the same as the sample size.

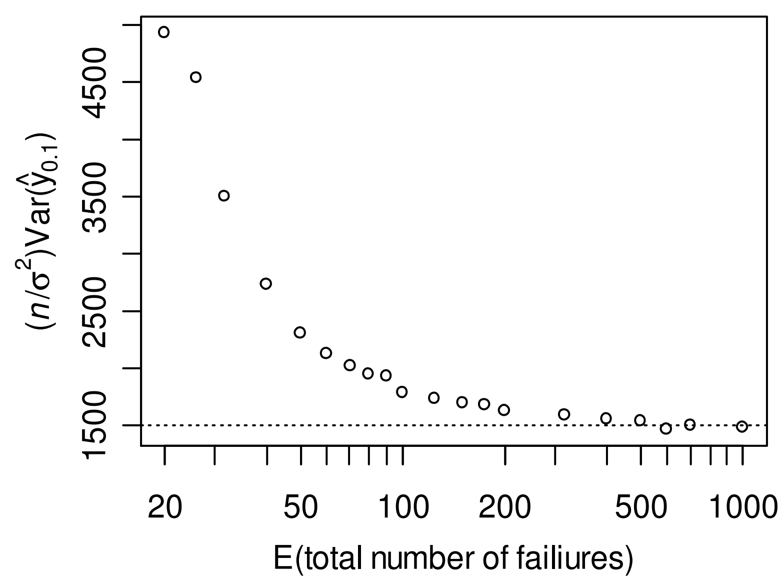

Figure 3.2: Simulated scaled variance as a function of the expected total number of failures under the two-dimensional optimum ramp-stress test plan for the Weibull distribution as compared to the corresponding large-sample approximate variance (horizontal dashed line). The number of simulations at

each point is 5,000. The optimum two-dimensional test plan has the following conditions: γ_0' = 6.0, γ_1' = 9.0, σ = 0.5, V_L = 13.9 kV, V_U = 20 kV, V_H = 40 kV, η = 2400 sec, *k* = 18.9 V/sec and *p* = 0.1.

3.4.3 Comparison of optimum ramp stress plans with optimum constant-stress plans

In [6], we compared the optimum step-stress and constant-stress test plans in terms of the large-sample approximate variance of the ML estimators of a quantile of interest. In this paper we extend the comparison to optimum ramp-stress tests. For the constant-stress tests the two-dimensional optimization is done by adjusting the lower level of stress as well as the allocations of test units to the constant stress levels. For step-stress tests, the optimization is done by adjusting the lower level of stress as well as the fraction of test time spent at each level. Note that a ramp-stress test plan is a limit of a multiple step-stress test plan under the limiting process described in Section 3.3.1. Note also that the optimum two-level step-stress has the smallest variance for reasonable planning values without any constraint on the fraction of time at each of stress levels. Thus, the variance of a ramp-stress test plan is not expected to be smaller than the variance of the optimum two-level step-stress test plan given the same planning values. Therefore, we compare the ramp-stress test plans with commonly used constant-stress test plans with the same planning values used in [6].

Figure 3.3 shows $\left(n/\sigma^2\right)\mathrm{Avar}\left(\hat{y}_p\right)$ as a function of σ under the optimum ramp-stress test plans with α = 2 (left) and α = 10 (right) for p = 0.01, 0.10 and 0.50, respectively, at p_H = 0.9 and p_U = 0.001. $\left(n/\sigma^2\right)\mathrm{Avar}\left(\hat{y}_p\right)$ usually decreases as σ increases. The variance at p = 0.5 is the largest, followed by the variance for p = 0.1 and then that at p = 0.01. These are similar to what were observed in [6] for step-up-stress test plans. Figure 3.3 also shows that the optimum variance with a larger α is slightly larger than that with a smaller α under the same planning values.

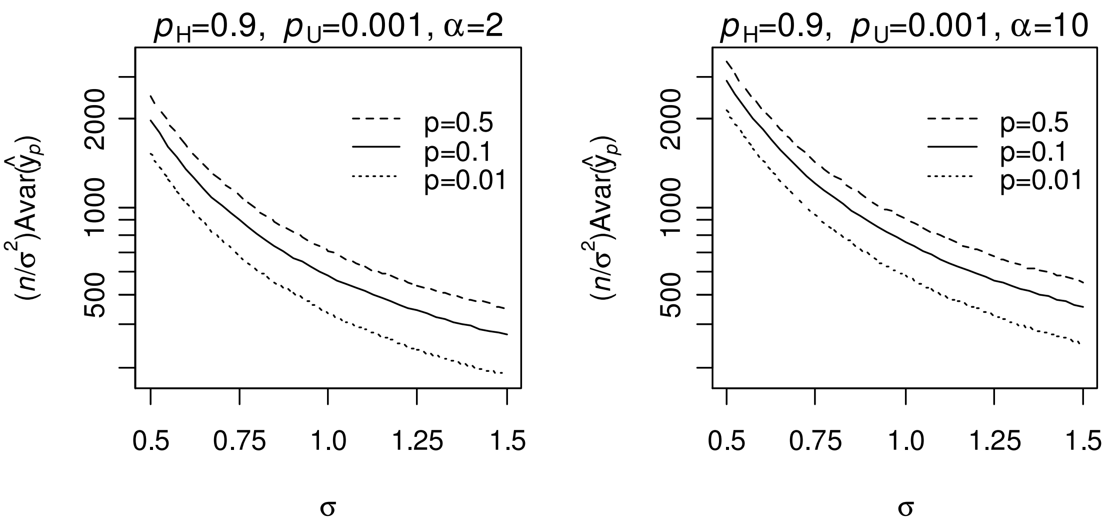

Figure 3.3: Scaled optimum variance as a function of σ for the Weibull distribution at three possible quantile of interest for p_H = 0.9 and p_U = 0.001 with α = 2 (left) and α = 10 (right). The three quantiles of interest are p = 0.01, 0.1, and 0.5.

Our major interest is variance comparison. In [6] the optimum two-level constant-stress $(n/\sigma^2)\text{Avar}(\hat{y}_p)$ are 95, 120 and 149 for p = 0.01, 0.10 and 0.50, respectively, at p_H = 0.9 and p_U = 0.001 under the Weibull distribution. Compared with the values of $(n/\sigma^2)\text{Avar}(\hat{y}_p)$ on the left of Figure 3.3, we find an optimum two-level constant-stress test plan has the smaller $(n/\sigma^2)\text{Avar}(\hat{y}_p)$ than that of an optimum ramp stress test plans at the situation we investigated. Our other results (not shown here) indicate that this conclusion also holds for the three other pairs of planning values (p_H = 0.9 and p_U = 0.00001), (p_H = 0.1 and p_U = 0.00001) and (p_H = 0.1 and p_U = 0.001) when σ is between 0.5 and 1.5.

3.5. Concluding Remarks

In this paper we propose an approach for computing the large-sample approximate variance of ML estimators of quantiles of the widely-used log-location-scale family of distributions with continuous time-varying stress accelerated life tests and censoring. We applied this approach to a situation that corresponds to ramp-stress sample data described in [3]. The results obtained by our more general method of computing large-sample approximate variances agree with the special-case results reported in [3]. We also extend previously-developed one-dimensional optimum ramp-stress test plans to the two-dimensional optimum ramp-stress test plan. The variance of the ML estimator of a quantile at the planning values was investigated for the Weibull distribution and was found to be larger for the two-dimensional optimum ramp-stress test plans than for the simple optimum two-level constant-stress test plans.

There are a number of possible extensions of the work presented here. These include

- The formulas in the appendix could be extended to multiple-stress situations (e.g., temperature and voltage) where either or both could be increasing in the test.
- We have considered only log-location scale distributions. It would be possible to derive similar results for other non-log-location-scale distributions (e.g., the gamma distribution).
- The response in an accelerated life test is time to failure. In some accelerated tests, the response is degradation. In some applications degradation can be monitored or measured periodically (see, for example, Meeker, Escobar, and Lu [9]). In other applications degradation is a destructive measurement (see, for example, Nelson [12] and Escobar, Meeker, Kugler and Kramer [5]). It would be interesting to develop methods parallel to ours where the degradation measure follows a log-location-scale distribution.

Appendix

This appendix provides the derivatives of the log likelihood given in Section 3.3 and the details of the approach that we used to calculate needed expectations and the large-sample approximate variance-covariance matrix of the ML estimators of γ_0, γ_1 and σ. For a log-location-scale distribution with multiple levels of stress, we have the following properties

$$\frac{\partial z_i}{\partial \gamma_0} = -\frac{1}{\sigma},\ \frac{\partial z_i}{\partial \sigma} = -\frac{z_i}{\sigma} \text{ and } \frac{\partial z_i}{\partial \gamma_1} = \frac{1}{\sigma}(\nu_i - \xi_i)\ ,$$

where $\nu_i = \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1})(\xi_i - \xi_j) e^{\gamma_1(\xi_i - \xi_j)} / (t - \delta_{i-1}),\ i = 2,3,\ldots,h$. Let $\nu_1(t) = 0$ and ν_η is ν_h at $t = \eta$. z_i is defined in Section 3.3.1.

Using the above properties, for a log-location-scale distribution, the first partial derivatives of the log likelihood in (1) with respect to parameters for a single observation can be expressed as

$$\frac{\partial l}{\partial \gamma_0} = -\frac{1}{\sigma}\left[\sum_{i=1}^{h} U_i(t)\frac{1}{\phi(z_i)} \cdot \frac{d\phi(z_i)}{dz} - U_{h+1}(t)\frac{1}{1-\Phi(z_\eta)} \cdot \frac{d\Phi(z_\eta)}{dz}\right]$$

$$\frac{\partial l}{\partial \gamma_1} = -\frac{1}{\sigma}\left\{\sum_{i=1}^{h} U_i(t)\left[\sigma\nu_i - \frac{1}{\phi(z_i)} \cdot \frac{d\phi(z_i)}{dz}(\nu_i - \xi_i)\right] + U_{h+1}(t)\frac{1}{1-\Phi(z_\eta)} \cdot \frac{d\Phi(z_\eta)}{dz}(\nu_\eta - \xi_h)\right\}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{1}{\sigma}\left\{\sum_{i=1}^{h} U_i(t)\left[1 + \frac{z_i}{\phi(z_i)} \cdot \frac{d\phi(z_i)}{dz}\right] - U_{h+1}(t)\frac{z_\eta}{1-\Phi(z_\eta)} \cdot \frac{d\Phi(z_\eta)}{dz}\right\}.$$

Define

$$A(z) = \frac{1}{\phi(z)} \cdot \frac{d^2\phi(z)}{dz^2} - \frac{1}{\phi^2(z)} \cdot \left(\frac{d\phi(z)}{dz}\right)^2,\ B(z) = \frac{1}{1-\Phi(z)} \cdot \frac{d^2\Phi(z)}{dz^2} + \frac{1}{[1-\Phi(z)]^2} \cdot \left(\frac{d\Phi(z)}{dz}\right)^2.$$

Note that the first partial derivatives are zero when evaluated at ML estimates when they are not on the boundary of the parameter space. Using this result, we set terms corresponding to the first partial derivatives equal to zero in the expressions for the second partial derivatives of the log likelihood. Then the resulting second partial derivatives of (1), evaluated at the ML estimates are:

$$\frac{\partial^2 l}{\partial \gamma_0{}^2} = \frac{1}{\sigma^2}\left[\sum_{i=1}^{h} U_i(t)A(z_i) - U_{h+1}(t)B(z_t)\right].$$

$$\frac{\partial^2 l}{\partial \gamma_1{}^2} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)\left[-\sigma^2 \nu_i' + \sigma \nu_i' \frac{1}{\phi(z_i)} \cdot \frac{d\phi(z_i)}{dz} + (\nu_i - \xi_i)^2 A(z_i)\right]\right\}$$

$$-\frac{U_{h+1}(t)}{\sigma^2}\left[\sigma \nu_\eta' \frac{1}{1-\Phi(z_\eta)} \cdot \frac{d\Phi(z_\eta)}{dz} + (\nu_\eta - \xi_h)^2 B(z_\eta)\right]$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)\left[-1 + z_i{}^2 A(z_i)\right] - U_{h+1}(t) z_\eta{}^2 B(z_\eta)\right\}$$

$$\frac{\partial^2 l}{\partial \sigma \partial \gamma_0} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t) z_i A(z_i) - U_{h+1}(t) z_\eta B(z_\eta)\right\}$$

$$\frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_0} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t) \cdot (\xi_i - \nu_i) A(z_i) + U_{h+1}(\nu_\eta - \xi_h) B(z_\eta)\right\}$$

$$\frac{\partial^2 l}{\partial \gamma_1 \partial \sigma} = \frac{1}{\sigma^2}\left\{\sum_{i=1}^{h} U_i(t)\left[-\sigma \nu_i - (\nu_i - \xi_i) \cdot z_i A(z_i)\right] + U_{h+1}(t)(\nu_\eta - \xi_h) \cdot z_\eta B(z_\eta)\right\},$$

where $\nu_i' = \partial \nu_i / \partial \gamma_1 = \sum_{j=1}^{i-1} (\tau_j - \tau_{j-1})(\xi_i - \xi_j)^2 e^{\gamma_1(\xi_i - \xi_j)} / (t - \delta_{i-1}) - \nu_i{}^2$, $\nu_\eta' = \nu_h'(\eta)$ and $i = 2,3,\ldots,h$.

In the limit, where $h \to \infty$ and the stress level change at each jump approaches to zero while holding the overall change of stress level constant, we have

$$z(t) = \left[\log\left(e^{\gamma_1 \xi(t)} w(t)\right) - \mu(\xi(t))\right]/\sigma = \left[\gamma_1 \xi(t) + \log(w(t)) - \mu(\xi(t))\right]/\sigma = \left[\log(w(t)) - \gamma_0\right]/\sigma$$

$dz(t) = d\left[\log(w(t)) - \gamma_0\right]/\sigma = q(t)\,dt/\sigma$, where $q(t) = e^{-\gamma_1 \xi(t)}/w(t)$.

Let $\theta = (\gamma_0, \gamma_1, \sigma)$. The Fisher information matrix of a time-varying stress ALT can be computed as $I_\theta = E\left[-\frac{\partial^2 l}{\partial\theta\,\partial\theta^T}\right]$. After factoring out the common factor of $1/\sigma^2$, the Fisher information matrix still depends on σ through $\partial^2 l/\partial\gamma_1{}^2$, $\partial^2 l/\partial\gamma_0\partial\gamma_1$ and $\partial^2 l/\partial\gamma_1\partial\sigma$.

Defining $C = \frac{1}{\phi(z)}\cdot\left(\frac{d\phi(z)}{dz}\right)^2 - \frac{d^2\phi(z)}{dz^2}$, $D = \frac{d\phi(z_\eta)}{dz} + \frac{\phi(z_\eta)^2}{1-\Phi(z_\eta)}$,

$R_m = \int_{-\infty}^{z_t} x^m C(x)dx$, $H_m = e^{-m\sigma z}C(z)$ and $Q_m = e^{-m\sigma z} zC(z)$ with m = 0, 1 or 2,

$G_1 = \left(\lambda e^{-\sigma z} - k^2 e^{-2\sigma z}\right)\phi(z)$, $G_2 = \left(\lambda e^{-\sigma z} - k^2 e^{-2\sigma z}\right)\frac{d\phi(z)}{dz}$,

one has

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_0{}^2}\right) = R_0 + D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_1{}^2}\right) = \int_0^\eta \left(\sigma^2 G_1 - \sigma G_2 + k^2 H_2 - 2k\xi H_1 + \xi^2 H_0\right)\frac{q}{\sigma}dt + \sigma\nu_\eta{}'\phi(z_\eta) + \left(\nu_\eta - \xi_h\right)^2 D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\sigma^2}\right) = \Phi(z_\eta) + R_2 + z_\eta{}^2 D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\sigma\,\partial\gamma_0}\right) = R_1 + z_\eta D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_1\partial\gamma_0}\right) = \int_0^\eta (\xi H_0 - kH_1)\frac{q}{\sigma}dt + \left(\xi_h - \nu_\eta\right)D$$

$$\sigma^2 \cdot E\left(-\frac{\partial^2 l}{\partial\gamma_1\partial\sigma}\right) = \int_0^\eta (\sigma\cdot\nu\cdot\phi(z) + (\xi-\nu)\cdot z\cdot C)\frac{q}{\sigma}dt + \left(\xi_h - \nu_\eta\right)z_\eta D$$

where $k(t) = \int_0^t (\xi(t)-\xi(x))\,e^{-\gamma_0-\gamma_1\xi(x)}dx$, $\lambda(t) = \int_0^t (\xi(t)-\xi(x))^2 e^{-\gamma_0-\gamma_1\xi(x)}dx$

$\nu(t) = e^{\gamma_0}k(t)/w(t)$, $\nu'(t) = e^{\gamma_0}\lambda(t)/w(t) - \nu^2(t)$.

In the case of the Weibull distribution, as used in our illustration in this paper, one could use the simplifications $C = e^{z}\phi_{\text{sev}}(z)$ and $D = \phi_{\text{sev}}(z_{\eta})$, where $\phi_{\text{sev}}(z) = \exp[z - \exp(z)]$ is the standard pdf of the smallest extreme value distribution.

Under the standard regularity conditions, the large-sample approximate variance-covariance matrix of the ML estimators of γ_0, γ_1 and σ is

$$\Sigma_{\hat{\gamma}_0, \hat{\gamma}_1, \hat{\sigma}} = \begin{bmatrix} A\,\mathrm{var}(\hat{\gamma}_0) & A\,\mathrm{cov}(\hat{\gamma}_0, \hat{\gamma}_1) & A\,\mathrm{cov}(\hat{\gamma}_0, \hat{\sigma}) \\ A\,\mathrm{cov}(\hat{\gamma}_0, \hat{\gamma}_1) & A\,\mathrm{var}(\hat{\gamma}_1) & A\,\mathrm{cov}(\hat{\gamma}_1, \hat{\sigma}) \\ A\,\mathrm{cov}(\hat{\gamma}_0, \hat{\sigma}) & A\,\mathrm{cov}(\hat{\gamma}_1, \hat{\sigma}) & A\,\mathrm{var}(\hat{\sigma}) \end{bmatrix}$$

$$= \frac{1}{n}\begin{bmatrix} E\left(-\dfrac{\partial^2 l}{\partial \gamma_0{}^2}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \gamma_1}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \sigma}\right) \\ E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \gamma_1}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_1{}^2}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_1 \partial \sigma}\right) \\ E\left(-\dfrac{\partial^2 l}{\partial \gamma_0 \partial \sigma}\right) & E\left(-\dfrac{\partial^2 l}{\partial \gamma_1 \partial \sigma}\right) & E\left(-\dfrac{\partial^2 l}{\partial \sigma^2}\right) \end{bmatrix}^{-1}.$$

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CHAPTER 4. STRATEGY FOR PLANNING ACCELERATED LIFE TESTS WITH SMALL SAMPLES SIZES

Haiming Ma and W. Q. Meeker
Department of Statistics
Iowa State University
Ames, IA 50011

Abstract

Previous work on planning accelerated life tests has been based on large-sample approximations to evaluate test plan properties. In this paper, we use more accurate simulation methods to investigate the properties of accelerated life tests with small sample sizes where large-sample approximations might not be expected to be adequate. These properties include the simulated s-bias and variance for quantiles of the failure-time distribution at use conditions. We focus on using these methods to find practical compromise test plans that use three levels of stress. We also study the effects of not having any failures at test conditions and the effect of using incorrect planning values. We note that the large-sample approximate variance is far from adequate when the probability of zero failures at certain test conditions is not negligible. We suggest a strategy to develop useful test plans using a small number of test units while meeting constraints on the estimation precision and on the probability that there will be zero failures at one or more of the test stress levels.

Key Words – Large-Sample Approximate Variance, Maximum likelihood, Reliability, Simulation.

Acronyms

ALT	accelerated life test
ZFP1	problem when zero failures occur at one or more levels of stress
ZFP2	problem when zero failures occur at two or more levels of stress
ML	maximum likelihood
CPPV	critical point planning values

Notation

n	total number of test units
s_U , s_H	pre-specified use level and highest level of stress
s_L, s_M	lowest and middle levels of stress
π_L, π_M , π_H	allocations of test units at s_L, s_M and s_H , respectively
ξ	standardized stress level $\xi = (s - s_U)/(s_H - s_U)$
t, η	failure time and censoring time
μ, σ	location and scale parameters of a location-scale distribution
γ_0, γ_1	parameters of the log-linear regression model
$\phi(\cdot)$, $\Phi(\cdot)$	standard pdf and cdf, respectively, of a location-scale distribution
p_U , p_L, p_M, p_H	probabilities that a unit will fail by time η at use, lowest, middle and highest stress levels, respectively
π_L^E , (ξ_L^E)	allocation (lowest level of stress), corresponding to having an equal expected number of failures at each of the three levels of stress for a fixed value of ξ_L (π_L).

π_L^{Opt}, (ξ_L^{Opt})	allocation and lowest level of stress, respectively, corresponding to overall optimum (minimum) variance of the quantile estimators obtained by adjusting π_L and ξ_L, simultaneously.
z_p	p quantile of a standard location-scale distribution
$y_p = y_p(\xi)$	p quantile of a location-scale distribution at stress level ξ

4.1 Introduction

4.1.1 Previous work

In an accelerated life test (ALT), units are tested at higher than usual levels of stress (e.g., temperature, voltage, or pressure) to obtain information about reliability in a small amount of time. ALTs are commonly used in product design and testing processes (see, for example, Chapter 6 of Nelson [11] and Chapters 18-20 of Meeker and Escobar [9]). Previous ALT planning methods have been based on large-sample approximations to assess test plan properties. The test plan properties (and corresponding approximations) depend on the model parameters. Thus one needs planning values for the parameters. As suggested in [5], the planning values can be given in terms of convenient quantities such as failure probabilities at the highest and use stress levels, respectively. As suggested in [12] and [13] information for planning values can be obtained from previous experience with similar products and materials or engineering judgment. Optimized two-stress-level test plans based upon such planning values that achieve the smallest large-sample approximate variance of the maximum likelihood (ML) estimators of interest (see, for example, ref. [9, 11 and 13]) have been studied extensively. To be robust to possible misspecification of the planning values and the relationship between the life and the levels of accelerating stress, compromise test plans with three or more levels have also been proposed and applied in practice [3, 7, 9 and 12].

4.1.2 Motivation

In practice, ALTs are usually subject to the constraint that the available number of test units has to be small either because of high cost of the units or availability of prototype units. In these cases, test planners may need to know the smallest possible number of units that are needed and how to choose the levels of stress and the allocation for those units to achieve a specified precision in the ML estimators.

We show how to find practical, statistically efficient constant-stress ALT plans with three levels of stress. When the sample sizes are small, test plans generated from large-sample approximations may not be adequate. In this paper, we use large-sample approximations for initial guidance but turn to simulation to do the needed evaluation of the properties of small-sample test plans that are needed to choose an actual plan. We illustrate the methods with an example. The results show that ALT test plans for small samples can be distinctly different from those suggested by large-sample approximations.

4.1.3 Overview

The remainder of this paper is organized as follows. Section 4.2 presents the model upon which our evaluations are based and introduces an ALT example that we use to illustrate how to evaluate the test-plan properties with small samples. Section 4.3 evaluates optimized compromise test plans with small samples. Section 4.4 studies test plans with the smallest zero failure probability and considers the impact of using incorrect planning values. Section 4.5 investigates the effect that using a small sample size will have on the adequacy of normal-approximation s-confidence intervals. Section 4.6 gives some concluding remarks and describes related areas for future research.

4.2 Model and ML Estimation

4.2.1 Setup

As described in [3] and [11], most ALT models require a transformation of stress (e.g., log of voltage). We use s to denote this transformed stress. All of the stress levels in the ALT will be between the use stress s_U and a pre-specified highest stress s_H. For convenience, we use the standardized stress $\xi = (s - s_U)/(s_H - s_U)$, where $s_U \leq s \leq s_H$ and $0 \leq \xi \leq 1$. Thus $\xi_U = 0$ and $\xi_H = 1$. All the test units are divided into three groups allocated at ξ_H , ξ_L, and ξ_M, respectively, where the middle level of stress is $\xi_M = (\xi_L + \xi_H)/2$. We assume, as is the case in most applications, that the three groups are tested simultaneously until a common censoring time η. With practical values of the planning values, if one does not use the kind of constraint suggested here (and in previous work with compromise ALT test plans), optimization results in an ALT plan with only two levels of stress (the optimum proportion at the middle level would approach zero or the optimum location of the middle level would approach one of the other two levels).

Constant-stress three-level compromise test plans can have a variety of forms. For example, Meeker and Escobar (see, Chapter 20 of [8]) suggest a compromise test plan with a fixed allocation proportion of 0.2 at ξ_M. In this paper, we modify this compromise test plan in the following way. Instead of a fixed π_M, the allocations π_L and π_M at ξ_L and ξ_M, respectively, are chosen such that the expected numbers of failures at ξ_L and ξ_M are equal. This modified compromise test plan is more appropriate for small sample sizes because it does a better job of controlling the probability of having zero failures at the lower stress levels. Under this constraint we choose ξ_L^{Opt} and π_L^{Opt} to obtain the optimized compromise test plan that minimizes the variance of the ML estimators of a specific function of the ALT model parameters. For a given compromise test plan, we obtain the exact (other than Monte Carlo error and conditioning on being able to estimate the model parameters) variances of the ML estimators by simulation and compare them with large-sample approximate variances. Our goal is to find an easy-to-apply method to choose a useful test plan defined by (π_L, ξ_L, n) that has good statistical properties and that can achieve the precision desired by a practitioner.

4.2.2 Model

Our assumed model corresponds to that used in most previous work in this area, summarized in Chapter 6 of [11] and Chapter 20 of [9]. At any level of the standardized stress ξ, the log failure time Y follows a location-scale distribution with constant σ and a cdf $\Pr(Y \le y) = \Phi[(y - \mu)/\sigma]$. The location parameter depends on (possibly transformed) stress through the linear relationship $\mu = \gamma_0 + \gamma_1 \xi$, where γ_0 and γ_1 are the regression model parameters. In our example, $\Phi(z) = \Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)]$ is the standardized smallest extreme value distribution corresponding to a Weibull failure time distribution. The failure probabilities at the highest stress and the use stress are $p_H = \Phi[(\log(\eta) - \gamma_0 - \gamma_1)/\sigma]$ and $p_U = \Phi[(\log(\eta) - \gamma_0)/\sigma]$, respectively. It is easy to express the probability at any other stress level ξ as a function of p_U and p_H. Given p_H, p_U, σ and η, one can easily calculate γ_0 and γ_1.

4.2.3 ML Estimation

Let $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{\sigma}$ denote the ML estimators of γ_0, γ_1, and σ, respectively. Then the ML estimator of the *p* quantile at stress level ξ can be expressed as $\hat{y}_p = \hat{\gamma}_0 + \hat{\gamma}_1\xi + z_p\hat{\sigma}$, where $z_p = \Phi^{-1}(p)$. The large-sample approximate variance of $\hat{y}_p$ is $\mathrm{Avar}(\hat{y}_p) = (1, \xi, z_p)\Sigma_{\hat{\gamma}_0,\hat{\gamma}_1,\hat{\sigma}}(1, \xi, z_p)$, where $\Sigma_{\hat{\gamma}_0,\hat{\gamma}_1,\hat{\sigma}}$ is the large-sample approximate variance-covariance matrix obtained from the inverse of the Fisher information matrix, and the superscript ^{T} indicates vector transpose (for details on how to do the computations, see, for example, Chapter 20 of Meeker and Escobar [9]). As is common practice in the ALT planning literature, one can compare the relative efficiency of test plans with different samples sizes using the scaled large-sample approximate variance denoted by $(n/\sigma^2)\mathrm{Avar}(\hat{y}_p)$. For small sample sizes, however, the scaled variance of $\hat{y}_p$ denoted by $(n/\sigma^2)\mathrm{Var}(\hat{y}_p)$ can be obtained to a much higher degree of approximation by using the Monte Carlos simulation, as described in detail in Section 4.2.4.

4.2.4 Zero failure problems

In ALTs with a fixed censoring time and small sample sizes, it is possible to have zero failures at one or more levels of stress at the end of the test. An ALT having zero failures at one or more levels of stress would generally be considered to be an unsuccessful ALT. Having zero failure at one or more levels of stress causes the loss of the advantages of a three-level test plan such as the ability to detect a departure from the assumed relationship between life and stress. We refer to this problem as the first type of zero failure problem (ZFP1). Having zero failures at two or more of the three levels of stress will make it impossible to estimate the model parameters or the quantile of interest. We refer to this problem as the second type of zero failure problem (ZFP2).

The s-bias and variance of $\hat{y}_p$ can be obtained via simulations in the following way. Based on the specified planning values, one can simulate sample ALT data. For each simulated data set, one can calculate the ML estimators $\hat{\gamma}_0$, $\hat{\gamma}_1$, and $\hat{\sigma}$ and $\hat{y}_p$. Using a large number of Monte-Carlo simulations, one can estimate $\mathrm{E}(\hat{y}_p)$ and $\mathrm{Var}(\hat{y}_p)$, conditional on no ZFP2 (because when a ZFP2 problem arises in a simulation trial estimation of the model parameters is not possible). Our simulation-based evaluations of $\mathrm{Var}(\hat{y}_p)$ are conditional on not having a ZFP2 [and thus, as we will see in our evaluations, such conditional variances could be misleading when Pr(ZFP2) is not negligible].

Generally, we want to find a test plan that has a small probability of having a ZFP1. Let n_L, n_M and $n_H = n - n_L - n_M$ be the number of test units allocated at ξ_L, ξ_M and ξ_H, respectively and let π_i denote the allocation at stress level ξ_i where $i = L$, M and H. The number of test units allocated to stress level i is $n_i = [\pi_i n]$, where n is the total sample size and [ ] means the rounding to the nearest integer, because $\pi_i n$ may not be an integer. The probability of failing at ξ_i is

$$p_i = \Phi\left(\frac{\log(\eta) - \gamma_0 - \gamma_1 \xi_i}{\sigma}\right). \tag{1}$$

The relationship between π_L and π_M in our compromise test plan is $\pi_L p_L = \pi_M p_M$, which implies an equal expected number of failures at ξ_L and ξ_M. Because all the test units are s-independent, the probability of having zero failures at ξ_i can be expressed as

$$P_i = (1 - p_i)^{n_i}, \tag{2}$$

where $i = L$, M and H. Thus the probability of ZFP1 is

$$\Pr(\mathrm{ZFP1}) = P_H + P_M + P_L - P_H P_L - P_M P_L - P_M P_H + P_H P_M P_L. \tag{3}$$

Similarly, the probability of ZFP2 can be expressed as

$$\Pr(\mathrm{ZFP2}) = P_H P_L + P_M P_L + P_M P_H - P_H P_M P_L \tag{4}$$

Given the levels of stress, the allocation corresponding to equal expected number of failures at each of the three stress levels π_i^E can be calculated by the following formulas

$$\pi_L^E = \frac{p_H p_M}{p_L p_M + p_M p_H + p_H p_L} \tag{5}$$

$$\pi_M^E = \frac{p_H p_L}{p_L p_M + p_M p_H + p_H p_L} \tag{6}$$

$$\pi_H^E = \frac{p_M p_L}{p_L p_M + p_M p_H + p_H p_L}. \tag{7}$$

Given the three allocations, one can also calculate the lowest and middle levels of stress, ξ_L^E and ξ_M^E, to have an equal expected number of failures at each of the three levels of stress. This can be done by noting that P_i, $i = L, M$ and H depend on the ξ_i values and thus one can solve the following equations for ξ_L^E and ξ_M^E:

$$\pi_H P_H = \pi_M P_M \tag{8}$$

$$\pi_M P_M = \pi_L P_L \tag{9}$$

Using (8) and the relationships (1) and (2), one can obtain ξ_M^E and ξ_L^E from (9).

From (3), one can obtain π_L^S, a value of π_L to minimize Pr(ZFP1). However, we do not have a simple analytical expression of π_L^S for given p_L, p_H and n. In practice, as is shown in Section 4.3.3, π_L^S is close to and a little larger than π_L^E. Because π_i^E $i = L, M$ and H do not depend on the sample sizes, using π_i^E can make the following test plan specification and evaluation simpler than using π_L^S .

4.2.5 The adhesive-bond ALT example and planning values

To illustrate the ideas presented in this paper, we will use the adhesive-bond test-planning example that was described in [7] and on page 535 of [9]. In this example, the

Weibull-distribution planning values were given as $p_H = 0.9$, $p_U = 0.001$, and $\sigma = 0.6$, based on previous experience with similar products. The censoring time is η = 183 days. The quantity of interest is the 0.1 quantile (i.e., p = 0.1) of the failure-time distribution at the use stress level of 50 ºC. Using the above planning values, we obtain γ_0 = 9.35, $\gamma_1 = -4.64$ and $y_{0.1}$ = 8.0 by using the formulas in Section 4.2.2. In the Meeker and Escobar [9] example, 300 units were available for testing. Here we will investigate test plans with fewer than 300 test units.

We will also assume there is uncertainty in the planning values. To do this we will evaluate the compromise test plan properties using alternative planning values. In Section 4.4.2 we show that when considering misspecification of the three planning values for the parameters over the range of a cube, it is sufficient to do one further evaluation at one of the corners of the cube. We call the planning values at this corner of the cube the *critical point planning values or* CPPV. For our example the CPPV is p_H = 0.45, p_U = 0.0005 and σ = 0.75.

4.3 Evaluation of Optimized Compromise Test Plan with Small Samples

4.3.1 Zero failure problems of a previous compromise test plan

It is important to investigate the zero-failure behavior of the compromise test plans such as those with a fixed π_M proposed in [6] and [9]. Fixing π_M = 0.2, one can obtain an optimized (i.e., minimum $\text{Avar}(\hat{y}_{0.1})$) compromise test plan by choosing π_L = 0.531 and ξ_L = 0.638 under planning values $p_H = 0.9$, $p_U = 0.001$, and $\sigma = 0.6$. Figure 4.1 shows the probabilities of both ZFP1 (dotted line) and ZFP2 (solid line) as a function of n when π_L =

0.531 and ξ_L = 0.638 for two points of planning values: $p_H = 0.9$, $p_U = 0.001$ and $\sigma = 0.6$ (on the left) and p_H = 0.45, $p_U = 0.0005$ and $\sigma = 0.75$ (on the right).

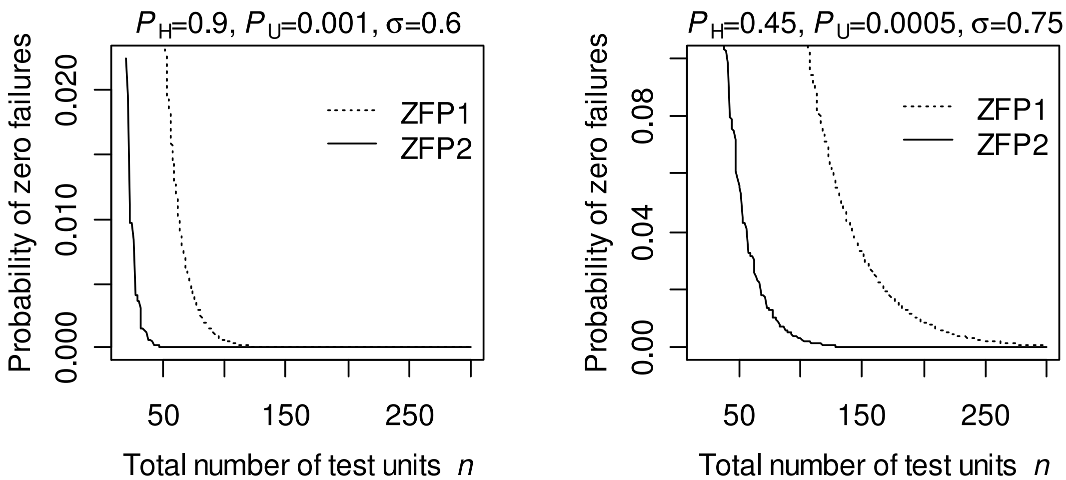

Figure 4.1: Pr(ZFP1) (dotted line) and Pr(ZFP2) (solid line) as a function of *n* when π_L = 0.531 and ξ_L = 0.638 for two sets of planning value(i) the original planning values: p_H = 0.9, p_U = 0.001 and σ = 0.6 (on the left) and (ii) the CPPV: p_H = 0.45, p_U = 0.0005 and σ = 0.75 (on the right).

Figure 4.1 shows that Pr(ZFP1) is much higher than Pr(ZFP2). This is because there are many more events leading to ZFP1 than ZFP2. Figure 4.1 also shows that the probabilities of having ZFP1 or ZPF2 under CPPV are much higher than the corresponding probabilities under the original planning values. Thus, it is important to consider the zero failure probabilities under both the original planning values and the corresponding CPPV.

To further investigate the cause of ZFP1 in Figure 4.1, consider $P_{LM} = P_L + P_M - P_L P_M$, the probability of having zero failures at either ξ_L or ξ_M regardless of the number of failures at ξ_H. If one plots P_{LM} versus *n* on Figure 4.1, the curve would indistinguishable from the curve of Pr(ZFP1) versus *n* in Figure 4.1. Thus ZFP1 is caused, primarily, by having no failures at either ξ_L or ξ_M. To assure that there are failures at both ξ_L and ξ_M, practitioners should control Pr(ZFP1) to be below some specified small value, say, 0.01.

It is possible that an optimized compromise test plan with π_M other than 0.2 may have a smaller probability of ZFP1 than that for π_M = 0.2. For each value of π_M, there is a corresponding optimized compromise test plan and a probability of ZFP1 associated with the plan. Figure 4.2 shows Pr(ZFP1) of those optimized compromise test plans as a function of π_M for n = 40, 60 and 90 and the original planning values (on the left) and the CPPV (on the right). The vertical dotted lines indicate the values of π_M that result in having an equal expected number of failures at ξ_L and ξ_M. The zigzag behavior comes from the integer sample-size rounding effect described in Section 4.2.4.

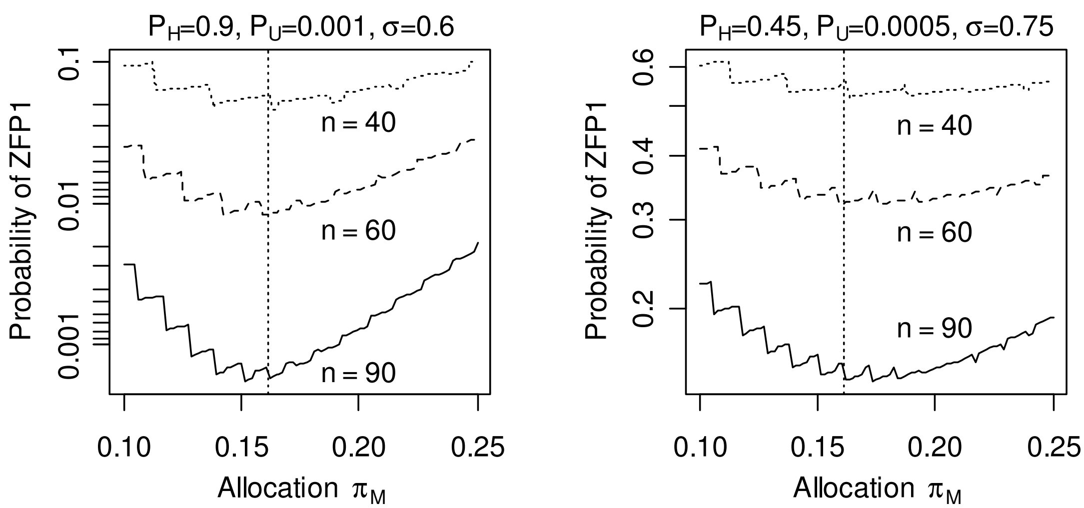

Figure 4.2: Pr(ZFP1) of optimized compromise test plans as a function of π_M for *n* = 40, 60 and 90 and two sets of planning values: (i) the original planning values (p_H = 0.9, p_U = 0.001 and σ = 0.6) on the left and (ii) the CPPV (p_H = 0.45, p_U = 0.0005 and σ = 0.75) on the right. The vertical dotted lines show the value of π_M at which the expected numbers of failures at ξ_L and ξ_M are equal.

Figure 4.2 shows that there is a value of π_M at which Pr(ZFP1) is minimum for a specific sample size. The value of π_M is close to that of an optimized compromise test plan with an equal expected number of failures at ξ_L and ξ_M [i.e., (9) holds]. Thus, to achieve, in a simple way, a small ZFP1 probability when the sample sizes are small, we suggest using a compromise test plan with an equal expected numbers of failures at ξ_L and ξ_M. In the remainder of this paper, the term compromise test plans refers only to the compromise test plans with an equal expected number of failure at ξ_L and ξ_M.

4.3.2 Adequacy of the large-sample approximate variance

Figure 4.3 shows the scaled actual variance $(n/\sigma^2)\mathrm{Var}(\hat{y}_{0.1})$ conditional on no ZFP2 (on the left) and the corresponding Pr(ZFP1) (on the right) as a function of the total sample size *n*, under the different planning values and test plans. The horizontal lines on the left plots show $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$. The actual variances were obtained by using Monte-Carlo simulations with data generated from the compromise test plan with an equal expected failure number at ξ_L and ξ_M described in Section 4.3.1. These figures provide an assessment of the adequacy of the large sample approximation for $(n/\sigma^2)\mathrm{Var}(\hat{y}_{0.1})$ and we can see when the approximation may be inadequate when n is too small.

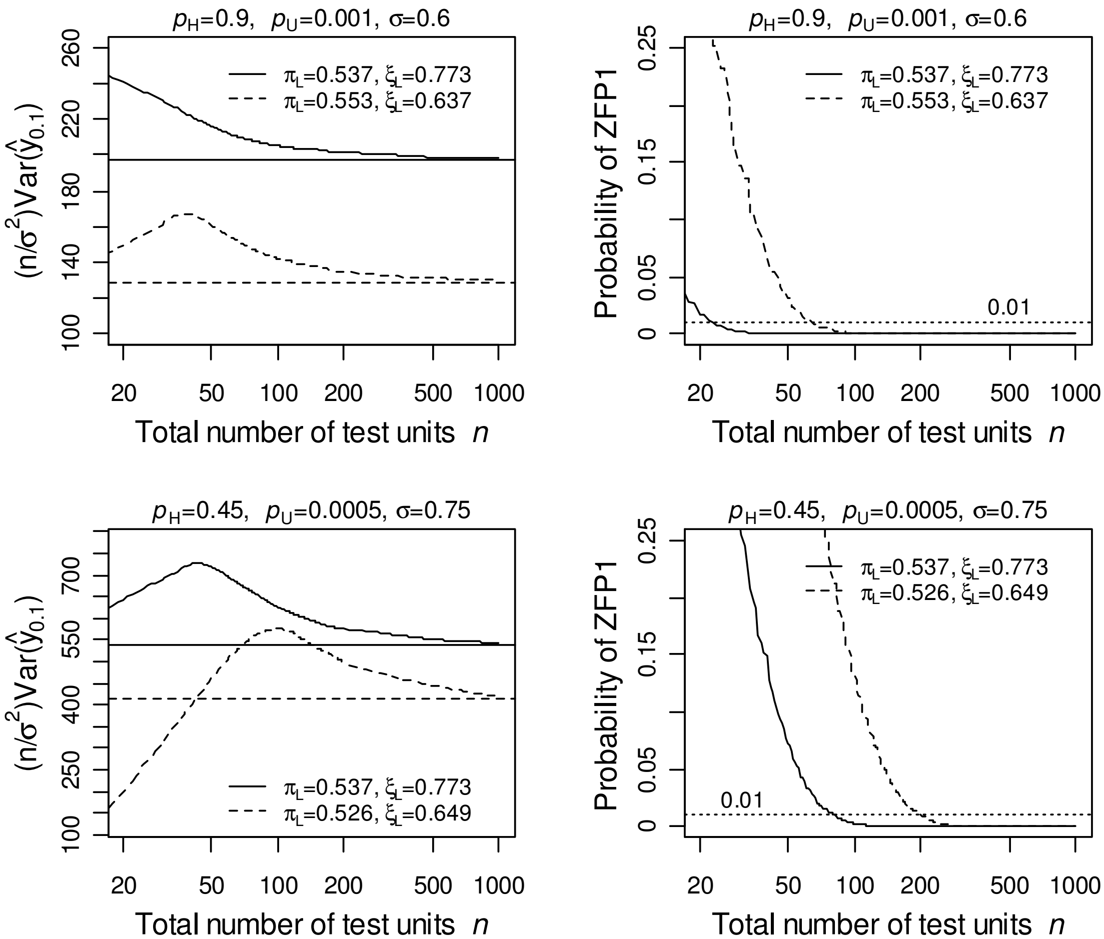

Figure 4.3: The plots on the left side are the smoothed scaled variances $(n/\sigma^2)\text{Var}(\hat{y}_{0.1})$, conditional on no ZFP2 as a function of *n* under original planning values (top) and the CPPV (bottom). The horizontal lines show the corresponding $(n/\sigma^2)\text{Avar}(\hat{y}_{0.1})$. The simulated curves are based on 10,000 simulations at each point using the Weibull distribution model under the compromise three–level constant-stress test plans with an equal expected number of failures at ξ_L and ξ_M. The plots on the right sideshow the corresponding Pr(ZFP1) calculated by using (3). The dotted horizontal lines indicate where Pr(ZFP1) = 0.01.

Figure 4.3 shows for the original planning values (top) and the CPPV (bottom), the optimized [i.e., the minimum $\text{Avar}(\hat{y}_{0.1})$] compromise test plans obtained by adjusting π_L and ξ_L. For the original planning values, the optimized compromise test plan has $\pi_L = 0.553$

and ξ_L = 0.637. For the CPPV planning values, the optimized compromise test plan has π_L = 0.526 and ξ_L = 0.649. As expected, given the same planning values and number of test units, compared with the non-optimized test plans, the optimized test plans provide smaller variances at the expense of a higher probability of ZFP1.

Another important observation from Figure 4.3 is that $(n/\sigma^2)\mathrm{Var}(\hat{y}_{0.1})$, conditional on no ZFP2, first increases with n until a maximum value and then decreases, approaching $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$ for large n. The maximum value of $(n/\sigma^2)\mathrm{Var}(\hat{y}_{0.1})$ can be as high as around 40% larger than $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$. The maximum value occurs when the probability of ZFP2 is between 0.01 and 0.02. The reason for this phenomenon is that when ZFP2 < 0.01, the probability of ZFP2 decreases rapidly with n, resulting in less conditioning and a more accurate representation of the true (unconditional) sample variability.

4.3.3 Reduction of the risk of ZFP1

Figure 4.4 shows Pr(ZFP1) as a function of the allocation π_L (left) or ξ_L (right) when ξ_L = 0.637 or π_L = 0.553, respectively. These probabilities were computed from the original planning values. Again, the zigzag behavior comes from the integer sample-size rounding effect described in Section 4.2.4.

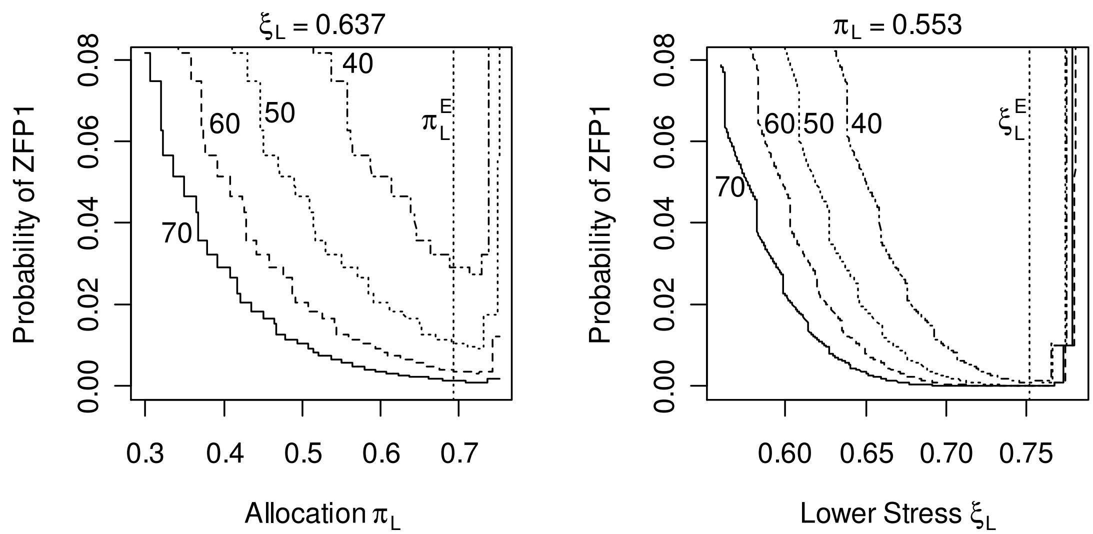

Figure 4.4: Pr(ZFP1) as a function of the unit allocation π_L (left) and the lowest level of stress ξ_L (right) at ξ_L = 0.637 or π_L = 0.553, respectively, for *n* = 40, 50, 60, and 70 with the original planning values p_H = 0.9, p_U = 0.001, and $\sigma = 0.6$. The dotted vertical lines correspond to π_L^E = 0.692 and ξ_L^E = 0.752, respectively.

In Figure 4.4, the left-hand plot shows that when π_L increases from $\pi_L = 0.3$, Pr(ZFP1) (primarily occurring at ξ_L or ξ_M in this situation) decreases until $\pi_L \approx \pi_L^E$. When π_L increases beyond π_L^E a value obtained from (5), Pr(ZFP1) (primarily occurring at ξ_H) will ultimately increase. The right-hand plot in Figure 4.4 shows that when ξ_L increases beyond ξ_L = 0.56, Pr(ZFP1) (primarily occurring at ξ_L or ξ_M in this situation) decreases until $\xi_L \approx \xi_L^E$, [where ξ_L^E is obtained by solving (8) and (9)]. When ξ_L increases beyond ξ_L^E, Pr(ZFP1) (primarily occurring at ξ_H) will, again, ultimately increase. To have more precise estimation with small sample sizes while controlling Pr(ZFP1) to be small, we suggest selecting π_L or ξ_L to be smaller than or close to π_L^E or ξ_L^E, respectively.

4.3.4 Evaluating the compromise test plan

It is interesting to examine the relationship between the simulated actual variance and the large sample approximate variance under the compromise test plans for finite sample sizes. We can compare the scaled variance of ML estimators as functions of π_L or ξ_L around the point ($\pi_L^{Opt} = 0.553$, $\xi_L^{Opt} = 0.637$), corresponding to the large-sample approximate optimized compromise test plan for the original planning values $p_H = 0.9$, $p_U = 0.001$, and $\sigma = 0.6$. Figure 4.5 shows $(n/\sigma^2)\mathrm{Var}(\hat{y}_{0.1})$ (solid and dotted curves) conditional on no ZFP2 and $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$ (dashed curves), both as a function of the allocation π_L with four different values of *n* and a fixed ξ_L. The parts of the curves with dotted lines in Figure 4.5 represent the values of π_L where Pr(ZFP1) ≥ 0.01. The $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$ curves are, of course, the same for all sample sizes. The two vertical dotted lines represent π_L^{Opt} (on the left) and π_L^{E} (on the right). These allocations can be calculated directly from (5), (8), and (9).

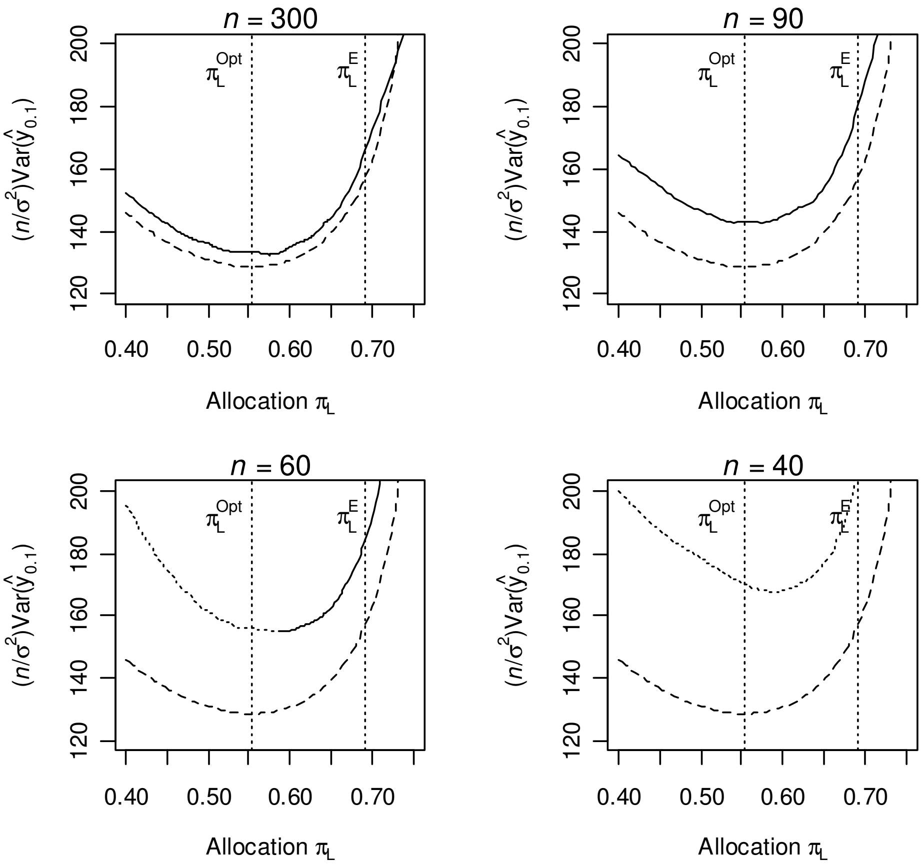

Figure 4.5: Smoothed (n/σ^2) Var$(\hat{y}_{0.1})$ conditional on no ZFP2 and (n/σ^2)Avar$(\hat{y}_{0.1})$ for the Weibull distribution failure-time model as a function of the allocation π_L based on 10,000 simulations at each point with ξ_L = 0.637 and the original planning values. The dashed lines show (n/σ^2)Avar$(\hat{y}_{0.1})$. The two vertical dotted lines represent π_L^{Opt} and π_L^{E}, respectively. The dotted parts of the smoothed curves correspond to the values of π_L where Pr(ZFP1) ≥ 0.01.

The simulated (n/σ^2) Var$(\hat{y}_{0.1})$ is larger than the large-sample approximate (n/σ^2)Avar$(\hat{y}_{0.1})$. When n = 300, 90, and 60 the minimum (n/σ^2) Var$(\hat{y}_{0.1})$ occurs at a value of π_L close to π_L^{Opt} [based on minimizing (n/σ^2)Avar$(\hat{y}_{0.1})$] . For n = 40 the minimum scaled variance is importantly larger than π_L^{Opt}. These results suggest that even when Avar$(\hat{y}_p)$ does not provide a good approximation for Var$(\hat{y}_p)$, it can provide a good approximation for minimizing (n/σ^2) Var$(\hat{y}_{0.1})$ as long as Pr(ZFP1) is not too large.

Another observation from Figure 4.5 is that, in the vicinity of π_L^E, $\left(n/\sigma^2\right)$ $\mathrm{Var}\left(\hat{y}_{0.1}\right)$ is smaller when $\pi_L < \pi_L{}^E$ than when $\pi_L > \pi_L^E$. When $\pi_L < \pi_L^E$, Pr(ZFP1) due to no failures at ξ_L or ξ_M is higher than that at ξ_H. When $\pi_L > \pi_L^E$, Pr(ZFP1) due to no failures at ξ_H is higher than that at ξ_L or ξ_M. Because there is a distinct increase in $\left(n/\sigma^2\right)$ $\mathrm{Var}\left(\hat{y}_{0.1}\right)$ at π_L^E when compared to that at π_L^{Opt}, as long as a specific criterion for the risk of ZFP1, say, Pr(ZFP1) $\leq$ 0.01, is satisfied, one should select a $\pi_L < \pi_L^E$ to reduce $\left(n/\sigma^2\right)$ $\mathrm{Var}\left(\hat{y}_{0.1}\right)$.

Figure 4.6 is similar to Figure 4.5, showing $\left(n/\sigma^2\right)$ $\mathrm{Var}\left(\hat{y}_{0.1}\right)$ and $\left(n/\sigma^2\right)\mathrm{Avar}\left(\hat{y}_{0.1}\right)$ as a function of ξ_L with four different values of *n* and a fixed value of π_L. Again, the dotted parts of the curves show where Pr(ZFP1) $\geq$ 0.01. The two vertical dotted lines indicate the location of ξ_L^{Opt} and ξ_L^E.

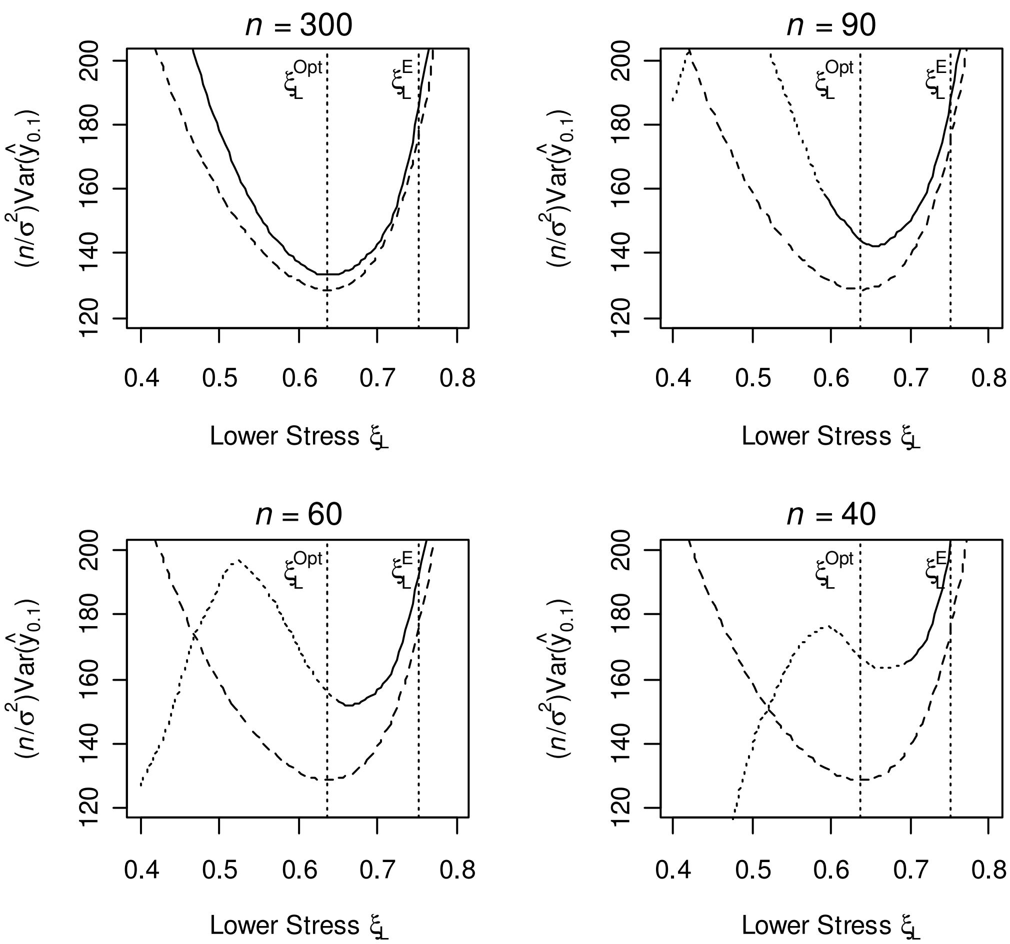

Figure 4.6: Smoothed (n/σ^2) $\mathrm{Var}(\hat{y}_{0.1})$ conditional on no ZFP2 for the 0.1 quantile (solid or dotted curves) of the Weibull failure distribution as a function of the lowest level of stress ξ_L from 10,000 simulations in each point with π_L = 0.553 and the planning values p_H = 0.9, p_U = 0.001 and σ = 0.6. The dashed lines represent $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$. The two vertical dashed lines represent ξ_L^{Opt} (the left) and ξ_L^{E} (the right). The dotted parts of the curves correspond to values of ξ_L where Pr(ZFP1) is larger than 0.01.

When n = 300, the minimum (n/σ^2) $\mathrm{Var}(\hat{y}_{0.1})$ occurs very close to ξ_L^{Opt}, the optimized lowest level of stress under the large-sample approximation. Note, however, that (n/σ^2) $\mathrm{Var}(\hat{y}_{0.1})$ is always larger than $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$ for n = 300. When n = 90, the simulated scaled variance (the dotted line) is increasing in ξ_L when ξ_L < 0.45. When ξ_L >

0.5, (n/σ^2) Var$(\hat{y}_{0.1})$ decreases in ξ_L until it reaches a minimum, after which it increases. When $n = 60$ or 40, the conditional (n/σ^2) Var$(\hat{y}_{0.1})$ is increasing in ξ_L when ξ_L is small and then is decreasing as ξ_L increases after a turning point. At the turning point the probability of ZFP2 is between 0.01 and 0.02, similar to the phenomenon described in Section 4.3.2. When ξ_L becomes larger, after passing through a minimum point, (n/σ^2) Var$(\hat{y}_{0.1})$ is increasing in ξ_L again. The values of ξ_L at the minimum point are a little larger than ξ_L^{Opt}. Note that the small values of the conditional (n/σ^2) Var$(\hat{y}_{0.1})$ are of little use when the probability of ZFP2 is importantly large (say greater than 0.01).

Another observation from Figure 4.6 is that, in the vicinity of ξ_L^E, (n/σ^2) Var$(\hat{y}_{0.1})$ is smaller when $\xi_L < \xi_L^E$ than when $\xi_L > \xi_L^E$. When $\xi_L < \xi_L^E$, Pr(ZFP1) is higher at ξ_L and ξ_M. When $\xi_L > \xi_L^E$, Pr(ZFP1) is higher at ξ_H. Because there is an important increase of the variance at ξ_L^E when compared to that at ξ_L^{Opt}, as long as a specific criterion to the risk of ZFP1 say, Pr(ZFP1) $\leq$ 0.01, is satisfied, one should select a $\xi_L < \xi_L^E$ to reduce the variance.

Figures 4.5 and 4.6 show that, when Pr(ZFP1) $\leq$ 0.01, although (n/σ^2) Var$(\hat{y}_{0.1}) \geq (n/\sigma^2)$Avar$(\hat{y}_{0.1})$, their minimum points in term of $(\pi_L,\ \xi_L)$ for different values of n are close to each other. This implies that the easy-to-compute (n/σ^2)Avar$(\hat{y}_{0.1})$ can be used as a guide to find an initial test plan.

Based on the information given in Section 4.3, we use the following strategy to find a useful ALT plan.

1. Use the simple analytical formulas in Section 4.2.4 to determine the region of $(\pi_L,\ \xi_L,\ n)$ in which Pr(ZFP1) is below the practitioner's ZFP1 critical level (say 0.01).
2. Minimize (n/σ^2)Avar$(\hat{y}_p)$, subject to the constraint that Pr(ZFP1) is less than the ZFP1 critical level, to obtain a tentative test plan.
3. Run simulations in the region of the tentative plan to fine tune the choice of π_L and ξ_L and to get the value of the actual variance for the test plan.

4.4 Test Plan Selection

4.4.1 Test plan properties for given planning values

Figure 4.7 is a contour plot showing $(n/\sigma^2)\text{Avar}(\hat{y}_{0.1})$ and Pr(ZFP1) as a function of ξ_L and π_L using the compromise three-level test plan described in Section 4.3.4. The contours show $(n/\sigma^2)\text{Avar}(\hat{y}_{0.1})$ and the zigzag parallel lines show Pr(ZFP1) for different sample sizes. Again, the zigzag behavior comes from the integer sample-size rounding effect described in Section 4.2.4. The solid line labeled $\pi_L^E \sim \xi_L^E$ shows where there is an equal expected number of failures at ξ_L and ξ_H. This line can be obtained by plotting π_L^E as a function of ξ_L, or equivalently by plotting ξ_L^E as a function of π_L. The region below this line is where we will find a useful test plan.

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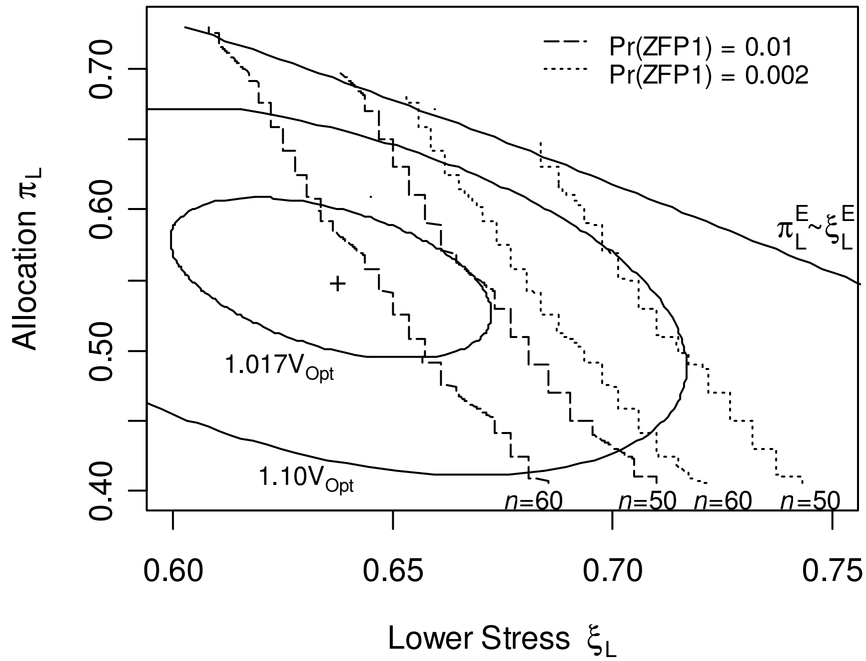

Figure 4.7: Contour plot showing $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$ and ZFP1 for the Weibull distribution model and the original planning values. The symbol "+" at the point (π_L = 0.553, ξ_L = 0.637) indicates the location of the minimum $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1}) = \mathrm{V}_{\mathrm{Opt}}$ = 128.7. The zigzag parallel lines correspond to particular values of Pr(ZFP1) (the dashed: 0.01 and the dotted: 0.002) for n = 50 and 60, respectively. The solid line labeled $\pi_L^E \sim \xi_L^E$ shows where the expected numbers of failures at ξ_L and ξ_H are equal.

Figure 4.7 illustrates the simple strategy to find a test plan. First, one can draw several zigzag lines representing a small ZFP1 probability, say 0.01, for different sample sizes. Then, draw the contours of the scaled large-sample approximate variance. Along a zigzag line (corresponding to a sample size and a ZPF1 constraint), a point that is close to a contour is a suitable candidate for the desired test plan having the smallest $\mathrm{Avar}(\hat{y}_p)$ for a specified sample size and small ZFP1 probability. Considering different zigzag lines, one can evaluate the tradeoff between sample size and Pr(ZFP1) to get the desired precision.

4.4.2 Test planning with uncertain of planning values

Because there is always some degree of misspecification in the planning values, it is important to check the impact that the uncertainty of planning values will have on the variance, the ZFP1 probability and the choice of test plan for small sample sizes. For the adhesive-bond example, if the practitioner is confident that the true values of p_H, p_U, and σ are in the intervals (0.45, 0.95), (0.0005, 0.0015), and (0.45, 0.75), respectively, based on previous experience, separate contour plots could be made for each combination and these could be used to find a plan that is satisfactory over the region of planning value uncertainty.

Suppose that the ranges of the planning-values uncertainty can be described by a cube containing all possible true values of (p_H, p_U, σ). There are eight corners in the cube. Note that $\mathrm{Avar}(\hat{y}_p)$ increases as σ increases and as p_H or p_U decreases, and the ZFP1 probability increases as p_H or p_U decreases. Therefore, the corner with the smallest p_H and p_U and the largest σ represents the largest possible values of both variance and the ZFP1 probability simultaneously. As described in Section 4.2.5, we refer to this combination as the critical planning value point (CPPV), because once the variance and Pr(ZFP1) at this set meet certain requirements, these requirements will be satisfied automatically throughout the entire cube. Thus, it is sufficient to investigate this critical point to evaluate the maximum impact of the incorrect planning values.

Figure 4.8 illustrates our procedure to find a good starting ALT test plan.

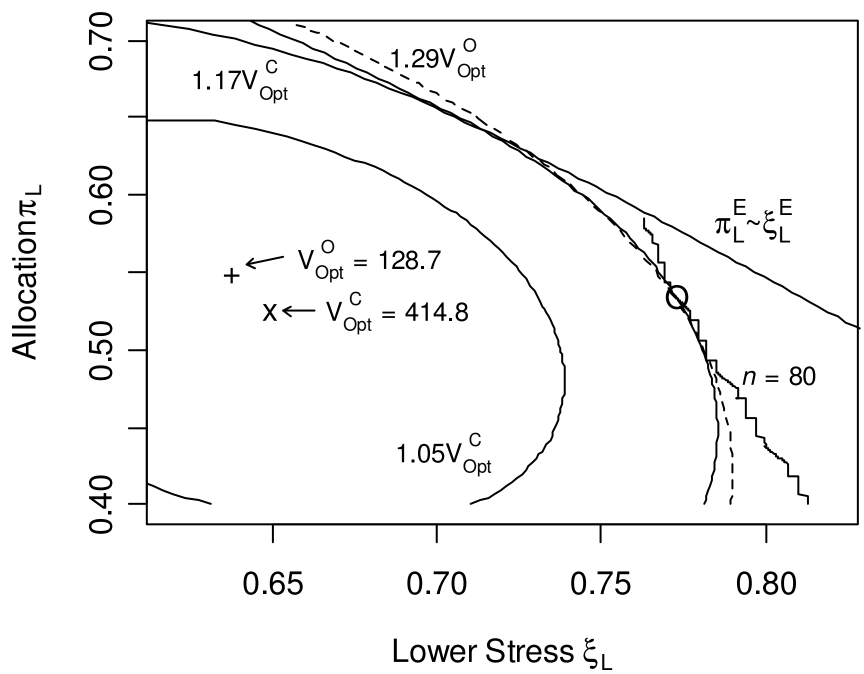

Figure 4.8: Contour plots illustrating the procedure to find a good starting ALT test plan (details in the text).

The contours in Figure 4.8 show $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$, relative to the value at minimum, when the true parameters are equal to the CPPV (solid lines) and the original planning values (dashed line). The "x" point indicates the position of the minimum point π_L = 0.526 and ξ_L = 0.649 where $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$ = 414.8 when the true parameters are equal to the CPPV. The "+" point indicates, as in Figure 4.7, the position of the minimum when the true parameters are equal to the original planning values. The solid curve labeled $\pi_L^E \sim \xi_L^E$ shows where there is an equal expected number of failures at ξ_L and ξ_H when the true parameters are equal to the CPPV. The zigzag line is where Pr(ZFP1) = 0.01 for n = 80. The dashed contour represents $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$, relative to the minimum when the true parameters are equal to the original planning values. The small circle indicates the point along the zigzag line where $(n/\sigma^2)\text{Avar}(\hat{y}_{0.1})$ is minimized and thus gives the tentative candidate test plan (π_L = 0.537, ξ_L = 0.773 and n = 80) when the true parameters are equal to the CPPV, thus taking into consideration the impact of the incorrect planning values. Note that Figure 4.3 shows (n/σ^2) $\text{Var}(\hat{y}_{0.1})$ conditional on no ZFP2 as a function of sample size when the test plans are chosen at the three points "x", "+" and the small circle.

Suppose that we desire to have Pr(ZFP1) $\leq$ 0.01 and $\text{Var}(\hat{y}_{0.1}) \leq 4.6$ at the CPPV. At the same time, the sample size should be as small as possible. Computing properties of the test plan at the small circle shows that, under the CPPV, $\text{Avar}(\hat{y}_{0.1})$ = 3.76. From Figure 4.3, there is roughly a 20% increase from $(n/\sigma^2)\text{Avar}(\hat{y}_{0.1})$ to $(n/\sigma^2)\text{Var}(\hat{y}_{0.1})$ when n = 80 and Pr(ZFP1) = 0.01 under the CPPV. Thus the actual $\text{Var}(\hat{y}_{0.1})$ under the CPPV will be approximately $1.2\times3.76=4.5$. Therefore, the test plan with π_L = 0.537, ξ_L = 0.773 and n = 80 is a candidate that meets our criteria at the CPPV. Using the same test plan with n = 80 and assuming that the values of the true parameters p_H, p_U, and σ are equal to the original planning values, we obtain $\text{Var}(\hat{y}_{0.1}) \approx 1.0$ using similar calculations.

4.4.3 Verification of the candidate test plan

Section 4.4.2 showed how to find a candidate test plan to control the Pr(ZFP1) and minimize (n/σ^2) $\text{Var}(\hat{y}_{0.1})$. Because there is uncertainty in the adequacy of the large sample approximate variance used in the initial optimization, however, the candidate test plan needs to be verified by more accurate simulations. To do this for the adhesive bond example, we examine the simulated scaled variance around the small circle along the zigzag line in Figure 4.8.

The solid lines in Figure 4.9 show the conditional (n/σ^2) $\mathrm{Var}(\hat{y}_{0.1})$ as a function of π_L when $n = 80$, corresponding to the point (π_L, ξ_L) on the zigzag line shown in Figure 4.8. The dashed zigzag curves show $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$. The plots on the left (right) are for the situation when the true parameters are equal to the original planning values (equal to the CPPV). The zigzag behavior of the curves is again the result of changing discrete allocations of test units to the stress levels. The location of π_L where the smallest (n/σ^2) $\mathrm{Var}(\hat{y}_{0.1})$ occurs under the CPPV is consistent with what is implied by Figure 4.8. In particular, along the zigzag line in Figure 4.8, the test plan around the point "O" has the smallest variance under the CPPV. The actual $\mathrm{Var}(\hat{y}_{0.1})$ values are 0.93 and 4.57 under the original planning values and the CPPV, respectively.

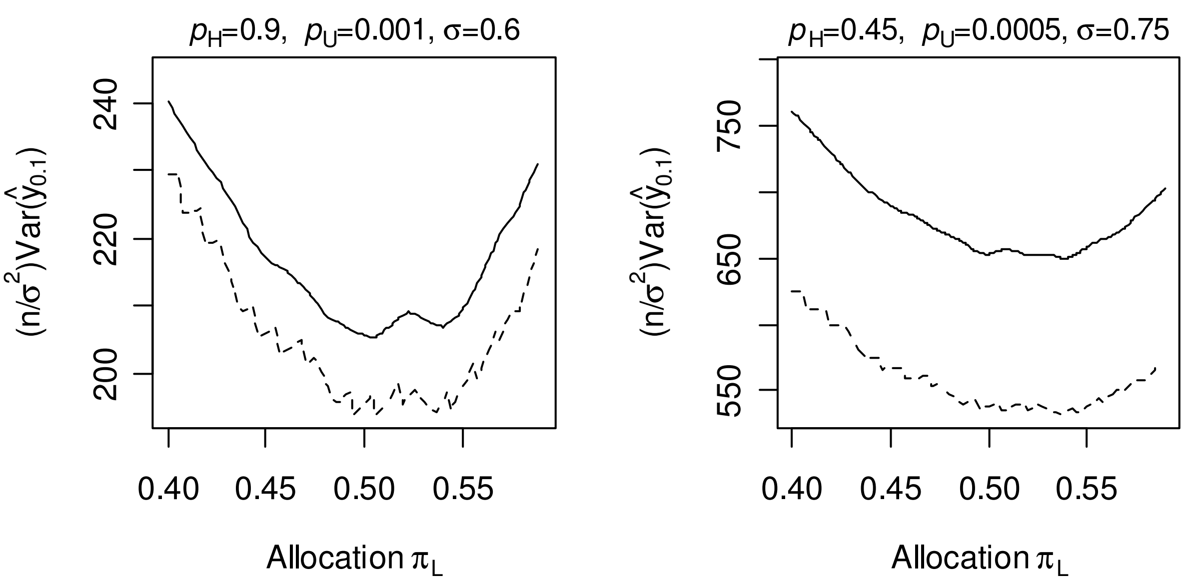

Figure 4.9: Simulated $(n/\sigma^2)\mathrm{Var}(\hat{y}_{0.1})$ (smoothed solid lines) conditional on no ZFP2 and $(n/\sigma^2)\mathrm{Avar}(\hat{y}_{0.1})$ (dashed lines) for the 0.1 quantile of the Weibull failure distribution as a function of π_L from 10,000 simulations at each point, while the point (π_L, ξ_L) is on the zigzag line shown in Figure 4.8 where Pr(ZFP1) = 0.01 with n = 80 under the CPPV. The left plot is for the original planning values

p_H = 0.9, p_U = 0.001 and σ = 0.6 and the right is for the CPPV p_H = 0.45, p_U = 0.0005 and σ = 0.75 .

Figures 4.8 and 4.9 illustrate the strategy to select a useful test plan that has a low risk of ZFP1 while achieving the smallest possible variance after considering the uncertainty of planning values. Note that constructing Figure 4.8 does not need any simulation. Thus this strategy minimizes the number of simulations that are needed and can allow one to find a useful test plan quickly.

Recall that Figure 4.2 shows that Pr(ZFP1) for a test plan with an equal expected number of failures at ξ_L and ξ_M is close to but may not be the minimum for a specific sample size due to the zigzag nature of Pr(ZFP1). Thus, it might be possible to find a slightly better test plan without the constraint of equal expected failure numbers at ξ_L and ξ_M around the small circle.

Finally, we would like to point out that the reason why we only consider the variance and not the s-bias of the quantile estimators is because, under the assumed model, the s-bias contribution to mean square error is negligible compared with variance when we control the risk of ZFP1 to be small.

4.5 Possible Departure from the Normal Approximation for Confidence Intervals

Normal approximation s-confidence intervals are based on the assumption that the quantity $\hat{z}_p = \left(\hat{y}_p - y_p\right) \big/ \sqrt{\widehat{\mathrm{Var}}\left(\hat{y}_p\right)}$ can be approximated by the standard normal distribution, where $\widehat{\mathrm{Var}}\left(\hat{y}_p\right)$ is usually the local-information estimator of $\mathrm{Var}\left(\hat{y}_p\right)$. We call $\hat{z}_p$ a "*t*-like" statistic because of its similarity to the *t*-statistic used in normal-distribution inference.

Especially when doing accelerated life testing with a small number of test units, the normal-distribution approximation may be inadequate when the expected number of failures is small. Here we show how to study the possible departure of actual coverage from the normal approximation.

Figure 4.10 shows normal Q-Q plots of $\hat{z}_{0.1}$ for n = 80, obtained from 1,000 simulations from the optimized compromise test plan (π_L = 0.553, ξ_L = 0.637) on the left, and the recommended test plan (π_L = 0.537, ξ_L = 0.773) on the right, for the true parameters p_H = 0.9, p_U = 0.001, σ = 0.6 (top), and the true parameters p_H = 0.45, p_U = 0.0005, σ = 0.75 (bottom), respectively. These two points in the parameter space correspond to the original planning values and the CPPV. Except for the plot on the NE of Figure 4.10, all of the plots show departures from the normal distribution in the upper tail. Interestingly, the departures are not too bad in the lower tail for the recommended plans on the NW and SE of Figure 4.10. Note, however, that a deviation in upper (lower) tail of the *t*-like statistics will lead to a lower (upper) s-confidence bound with poor coverage properties. In reliability applications, it is usually the lower bound on a quantile that is of most interest.

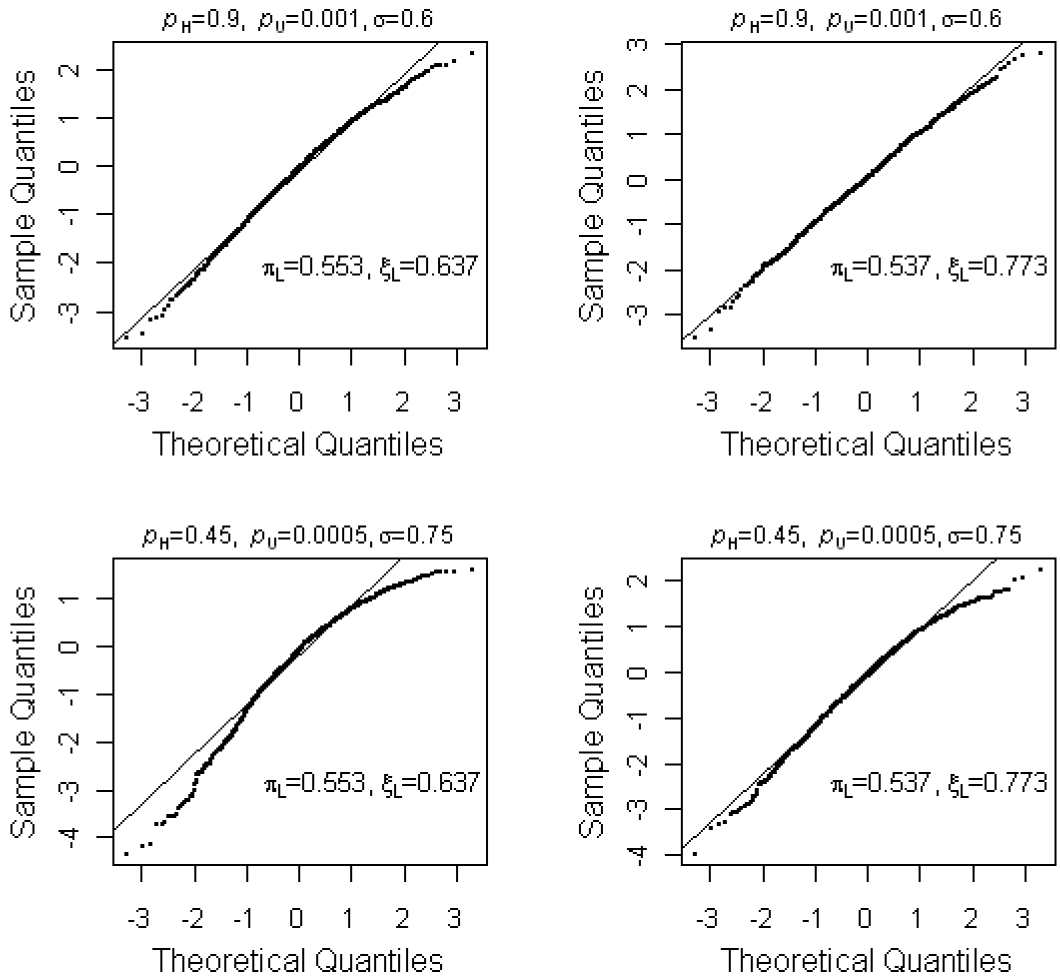

Figure 4.10: Normal Q-Q plots of $\hat{z}_{0.1}$ the simulated standardized variance conditional on no ZFP2 of the 0.1 quantile of the Weibull failure time distribution for n = 80, obtained from 1,000 simulations at test plans (π_L = 0.553, ξ_L = 0.637) on the left and (π_L = 0.537, ξ_L = 0.773) on the right for the true parameters to be the original planning values (top) and the CPPV (bottom), respectively.

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Table 4.1 shows the expected number failing at each test condition for each combination of test plan and planning values. As suggested by Table 4.1, the normal approximation tends to be especially poor when the expected number of failures at the individual test conditions is small. These results suggest that when the expected number of failures is small, one should use better s-confidence interval procedures such as those based on the bootstrap (see for example [4]) or the inversion of a likelihood ratio test (see for example [14]) to have a procedure with a more accurate coverage probabilities.

Table 4.1: Expected numbers of failures for different situations when the sample size is 80.

True Parameters	Test Plan	Figure 4.10	Expected Numbers of Failures			
			ξ_L	ξ_M	ξ_H	Total
Original planning values	Optimized	NW	5.7	5.7	20.2	31.6
Original planning values	Recommended	NE	14.1	14.1	12.7	40.9
CPPV	Optimized	SW	2.0	2.0	10.3	14.3
CPPV	Recommended	SE	4.8	4.8	7.4	17.0

4.6 Concluding Remarks and Areas for Future Research

In this paper, we address the issues involved in planning ALTs with small sample sizes. We describe and investigate the important role that the possibility of zero failures can have on the conditional variance. For constant 3-level ALT plans, using a compromise test plan with an equal expected number of failures at the lowest and middle levels of stress can reduce the ZFP1 probability so that smaller sample sizes become possible for a specified estimation precision and set of planning values. Furthermore, by using the plots of test plans such as those shown in Figure 4.8, one can select a tentative test plan without having to run time-consuming simulations. Then the tentative test plan needs to be fine-turned and verified by simulations. Finally, one needs to check whether the commonly-used normal approximation for s-confidence intervals provides an adequate approximation or not.

Due to the small sample sizes involved, there is not a simple theory to provide the actual variance over a large parameter range, as provided by the large sample approximations. However, one may use the strategy outlined in this paper to evaluate and find a good ALT test plan using simulations. Applying the ideas in this paper to other models and distributions should be straightforward.

In this paper we show how to construct a three-level compromise constant-stress test plan with small sample sizes. If using the smallest number of test units is a primary concern, one might want to use a simple two-level test plan. The planning methods for the three-level constant-stress test plan with small sample sizes can also be used to find a two–level constant-stress test plan with a small sample size. Simulations (for example, [4] and [14]) have shown that the adequacy of large sample approximation is closely related to the expected number of failures. Finally, we point out that it may be more appropriate to replace the term “small samples“ used in this paper as “small expected numbers of failures,” because under certain planning values, the expected number of failures is often small even if large numbers of test units are used.

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CHAPTER 5. CONCLUSION

ALTs are widely used in industry. Good ALT plans can yield significant benefits to industry. In this dissertation, we make two extensions to the previous research on the design of ALTs. One is proposing an approach for computing the large-sample approximate variance of ML estimators of quantiles of a general log-location-scale family of distribution with step-stress or continuous time-varying stress accelerated life tests and censoring. We then apply this approach to the design of step-stress and ram-stress accelerated tests. The other is proposing an approach for designing constant-stress ALTs with small sample sizes. Both extensions originated from the needs of engineers in their practical application of ALTs.

Our first extension was used to compare the large-sample approximate variances obtained from constant-stress, step-stress and ramp-stress ALT plans. Our results show that when σ, the scale parameter of the assumed log-location scale distribution, is small, optimum constant-stress ALT plans tend to have smaller variance than that of the corresponding optimum step-stress ALT plans. Besides this, constant-stress ALT plans possess the following advantages. Optimum constant-stress ALT plans usually tend to have larger numbers of failures than that of optimum step-stress ALT plans with the same sample size and planning values. Thus, the large-sample approximations can be expected to provide a better approximation for constant-stress ALT plans as compared with step-stress ALT plans. Moreover, because of the availability of appropriate procedures in commercial software packages, data from constant-stress ALTs are relatively easier to analyze.

Step-stress ALT plans tend to have a smaller variance than that of the corresponding optimum constant-stress ALT plans when σ, the scale parameters of the assumed log-location scale distribution, is large. Also, step-stress/ramp-stress ALTs have the following advantages.

- A step-stress/ramp-stress ALT only requires a single temperature-control chamber for testing.
- A step-stress/ramp-stress ALT is more flexible than a constant-stress ALT because when practitioners find that a step-stress ALT plan is not working as expected based on the given planning values, they can modify the original plan during the test (e.g., by increasing the rate of increase in a ramp-stress test).

A continuously varying-stress test plan can be treated as a limit of a step-stress test plan when the number of the step approaches to infinity while holding the overall change of stress level constant. Combining the formulas in Chapters 2 and 3, it becomes possible to calculate the large-sample approximate variance for a test plan under a general log location-scale distribution with a stress level varying in a general way.

The approach for computing the large-sample approximate variance is based on a general log location-scale distribution and a time-varying stress. Although in Chapters 2 and 3, we present applications of this approach and found it worked well within the range of the distributions and the planning values we investigated, this approach may be applied in the areas beyond the types of the ALT plans considered in this dissertation (e.g., test plans with a periodic stress or with two or more stress variables). With this approach, one does not have to develop a specific method to calculate the large-sample approximate variance for an ALT under a specific log location-scale distribution.

The second major thrust of this dissertation aims to solve an important practical problem: how to plan ALTs with small sample sizes. By examining the behavior of the variance of an ML estimator for a Weibull distribution quantile from small to large sample sizes, we find that the probabilities of zero failures play an important role and have a strong relationship with the conditional variance (conditional on having data that is sufficient to estimate the model parameters). For constant-stress 3-level ALT plans, using a compromise test plan with an equal expected number of failures at the lowest and middle levels of stress can reduce the probability of having zero failures at one or more of the test-stress levels, so that smaller sample sizes become possible for a specified estimation precision and set of planning values. When we control the probability of zero failures at one or more of the test-stress levels to be small, we find that one can use the large-sample approximation for initial exploration and optimization, without having to run time-consuming simulations. The tentative test plan does, however, need to be fine-turned and verified by simulations. This approach provides a useful strategy for planning ALTs with small sample sizes.

Other extensions to the previous research on the designs of ALTs are also of interest. For example, engineers would like to know how to plan ALTs under the uncertainly of the underlying distribution or in the underlying life-stress relationship. This will be a subject for our future research.